Basics of Nonlinear Plasma Theory

→ Introduction to Fluid Turbulence (NSE)
  ⓐ → homogeneous – Kolmogorov → $\boxed{\text{cascade}}$ → l
  ⓑ → pipe flow – Prandtl → $\boxed{\text{mixing}}$ – x

Local Interaction

→ Introduction to Wave Kinetics and Langmuir Turbulence
  ⓐ → Wave Kinetics → Whitham variational formulation (phase symmetry)
    → Boltzmann analogy
  ⓑ → Eikonal Theory of Langmuir Turbulence
    → Zakharov Eqns. + NLS
    → Self-Focusing + Collapse

Non-Local Interaction

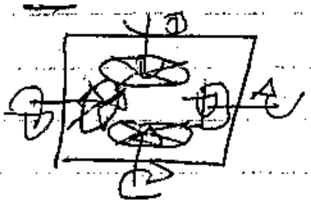
→ NL Wave-Particle Interaction – Intro
  ⓐ → Rev QLT / Kubo # + validity
  ⓑ → Higher order QLT
  ⓒ → Renormalization in Vlasov Turbulence.

Turbulence Theory

— An Introduction

P. Diamond

i.e.



I.) Basics of Fluid Turbulence (3D)

.) Characteristics of Fluid Turbulence:

"turbulence" vs "noise" → $\boxed{\text{energy flux}}$

- broad range of spatio-temporal scales excited — contrast TAM

Ref. U. Frisch "Turbulence - The Legacy of A.N. Kolmogorov"

- decay of large scale energy ⟹ need input/stirring to maintain stationarity
- energy input dissipated as heat (to maintain stationarity) → viscosity → irreversibility
- irreversible mixing occurs ⟹ i.e. passive tracer
- intermittancy manifested
  i.e. spatial ⟹ coherent structures (i.e. vortices) occur
  temporal ⟹ bursts — probe trace

- self-similarity / scale-similarity:

  turbulence "looks the same" on all scales, except the very largest (stirring) and the very smallest (dissipation)
  Caveat: Intermittancy - memory of large scales on small.

ii) Navier Stokes Equation – $\left\{\begin{array}{l}\text{Describes}\\ \text{Fluid}\end{array}\right.$

$$\underbrace{\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V}}_{\substack{\text{advection/}\\ \text{straining}\\ \to\ \text{nonlinearity}}} = \underbrace{-\frac{\underline{\nabla} p}{\rho}}_{\text{pressure}} + \underbrace{\nu \nabla^2 \underline{V}}_{\substack{\text{viscous}\\ \text{diffusion of}\\ \text{momentum}}}$$

$\rho = 1$ hereafter

$$\underline{\nabla}\cdot\underline{V} = 0 \qquad \text{incompressibility}$$

Note: Pressure determined from incompressibility

i.e.

$$\nabla\cdot\left[\cancel{\frac{\partial \underline{V}}{\partial t}} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right] = -\nabla^2 p + \nu\nabla^2\cancel{(\underline{\nabla}\cdot\underline{V})}$$

$$\nabla^2 p = -\nabla\cdot[\underline{V}\cdot\underline{\nabla}\,\underline{V}]$$

$$p = -\nabla^{-2}[\nabla\cdot\underline{V}\cdot\underline{\nabla}\,\underline{V}]$$

$$= \frac{1}{4\pi}\int \frac{d^3x'\,\underline{\nabla}'\cdot[\underline{V}(x')\cdot\nabla_{x'}\underline{V}(x')]}{|\underline{x}-\underline{x}'|}$$

More generally, can eliminate p

$$\partial_t V_i + (\delta_{i\ell} - \partial_{i\ell}\nabla^{-2})\,\partial_j(V_j V_\ell) = \nu\nabla^2 V_i$$

Key Parameter: Reynolds #

$$\boxed{\begin{aligned} Re &= |\underline{V}\cdot\nabla\underline{V}| \,/\, |\nu\nabla^2\underline{V}| \\ &\sim \frac{V(L)\,L}{\nu} \end{aligned}}$$

$\sim \dfrac{\text{nonlinearity}}{\text{collisional diffusion}}$

measure of strength of NL.

- Re usually referenced to largest scale

  $L = L_{max}$
  $V(L)$ = large-scale velocity

- Re always referenced to a particular scale

  L_{max}, $\lambda = \left[\langle(\partial_i V_j)^2\rangle / \langle V_i^2\rangle\right]^{-1/2}$, $\lambda_{dissn.}$,
  (Taylor Scale) (Re = 1)

- Re >> 1 in turbulent {pipe flow, atmosphere, etc.}

  $Re \sim 10^6 - 10^8$, etc.

  i.e. planetary boundary layer : $l_{out} \sim 1\,km$
  $\sim 10^5\,cm$
  $l_{diss} \sim .1\,cm$

  ⇒ 6 decades!.

- Re : measure of ratio of inertial mixing of momentum to collisional mixing...

iii) Experimental 'Laws' of Fully Developed Turbulence

→ Much / most of turbulence theory is empirically motivated. Experimental info/results preceded sophisticated theoretical analyses...

The experimental facts:

1.) 2/3 Law (Mundane)

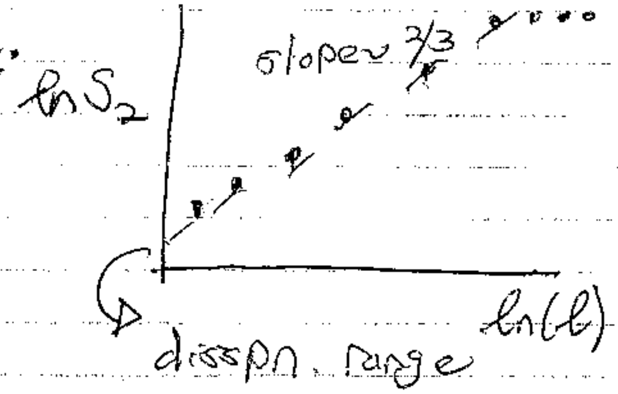
In a turbulent flow with $Re \gg 1$, $\langle \delta V(\ell)^2 \rangle$ (mean square velocity increment between two scales separated by distance ℓ) scales as $\ell^{2/3}$.

i.e. $\delta V(\ell) = |\underline{V}(\underline{r}+\underline{\ell}) - \underline{V}(\underline{r})|$ ⟹ {a difference!

$S_2(\ell) = \langle \delta V(\ell)^2 \rangle \sim \ell^{2/3}$ → fundamental scaling relation

S_2: 2nd order structure function

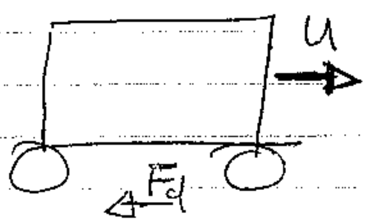
i.e.



B) Law of Finite Energy Dissipation (Profound)

If, in an experiment on turbulent flow, all the control parameters are kept the same, except the viscosity, which is lowered as much as possible, the energy dissipation per unit mass dE/dt behaves in a way consistent with a finite limit.

- What means 'Energy Dissipation Rate'?



Consider a car, experiencing atmospheric drag

$$F_d = \frac{1}{2} C_D \rho S u^2$$

S: face surface area

i.e.

$$P = \rho S (u\tau) u$$

→ momentum in air of slug:
$M = \rho S u \tau$ → mass
$V = u$

if assume air momentum completely transferred to car

$$\frac{dP_{car}}{dt} = F_d = \rho S u^2$$

$\frac{C_D}{2}(Re) \equiv$ drag coefficient (slowly varying function of Re, depends on shape, etc.)

$\therefore \quad F_d = \frac{C_D}{2} \rho S U^2$

Now, power dissipated by drag force

$$P_d = F_d U$$

$$\Rightarrow \quad P_d = \frac{C_D}{2} \rho S U^3$$

Energy dissipation rate (per volume) $\epsilon = P_d / \text{Mass}$

$$= \frac{C_D}{2} \frac{U^3}{L}$$

$$\sim U^3/L$$

also NS $\Rightarrow \partial_t \langle v^2 \rangle \sim -\langle \underline{v} \cdot \underline{\nabla} \, \underline{v}^2 \rangle$

$\Rightarrow$ Why should we care?

Note, energy budget:

$$\frac{\partial}{\partial t} V_i + V_j \partial_j V_i - \nu \nabla^2 V_i = -\partial_i p$$

$$\partial_t \frac{v_i^2}{2} + \partial_j \frac{v_j v_i^2}{2} - \nu v_i \nabla^2 v_i = -v_i \partial_i p$$

$\langle \ \rangle \equiv$ ensemble (fast space-time avg.)

$$\partial_t \left\langle \frac{v_i^2}{2} \right\rangle + \cancel{\left\langle \partial_j \frac{v_j v_i^2}{2} \right\rangle} - \nu \langle v_i \nabla^2 v_i \rangle$$

surface terms $= \cancel{\langle \partial_i v_i p \rangle}$ upon IBP

$\Rightarrow$

$$\partial_t \left\langle \frac{v_i^2}{2} \right\rangle = -\nu \langle |\nabla v_i|^2 \rangle$$

but $\epsilon \equiv -\partial_t \left\langle \frac{v_i^2}{2} \right\rangle$! (- dissp'n rate)

$\therefore$

$$\boxed{\epsilon = \nu \langle |\nabla v_i|^2 \rangle} \; !$$

$\Rightarrow$ experiments suggest that $\epsilon \to$ finite as $\nu \to 0$!!! $\Leftrightarrow$ re-markable

$\Leftrightarrow$ suggests that extremely large $\nabla \underline{v}$ forms as $\nu \to 0$, singular vortex sheets

$\Rightarrow$ singular velocity gradients formed in limit of weak viscosity !!

{ Heart of turbulence problem is grappling with singularity (especially its degree) of velocity gradients.

N.B.: $\left\{\begin{array}{l}\text{Dissipation Law} \\ \text{Singularity formation}\end{array}\right.$ is at the heart of why turbulence is a "hard" problem.

Re: Dissipation Law:

$$\epsilon \sim \frac{u^3}{L} \sim u^2/(L/u)$$

$$\sim \frac{\text{K.E. per Mass}}{\text{Circulation Time}}$$

i.e. → in 1 macro circulation time, a finite fraction of (macro) kinetic energy is dissipated by viscosity.

→ dissipation time scale is (L/u).

V.) Kolmogorov's Hypotheses and their Predictions/Implications. → K41 Theory of Turbulence

1: In the limit of $Re \to \infty$, all possible symmetries of the Navier-Stokes equation, usually broken by the mechanisms producing the turbulent flow, are restored in a statistical sense at small scales and away from boundarys.

lose memory

What means?

- "small scales" : $l \ll l_0$

l_0: integral scale → characteristic of production.

- symmetries

First, symmetries of Navier-Stokes Eqn. !!

a.) space translations $\underline{r} \to \underline{r} + \underline{\rho}$
(no explicit $\underline{r}$ dep.)

b.) time translation $t \to t + \tau$
(no t dep.)

* c.) Galilean boosts $\begin{cases} \underline{r} \to \underline{r} - \underline{u}t \\ \underline{V} \to \underline{V} + \underline{u} \end{cases}$ $\underline{V} = \underline{u} + \underline{V}(\underline{r} - \underline{u}t, t)$
(no frame dep.)

i.e. $\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} = -\nabla P + \nu \nabla^2 \underline{V}$

insert ⇒
$$\cancel{-\underline{u} \cdot \nabla \underline{V}} + \cancel{\underline{u} \cdot \nabla \underline{V}} + \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} = -\nabla P + \nu \nabla^2 \underline{V}$$

d.) Parity left-right $\underline{x} \to -\underline{x}$, $\underline{V} \to -\underline{V}$
(no preferred direction)

e.) Rotation $\begin{cases} \underline{r} \to R\,\underline{r} \\ \underline{V} \to R\,\underline{V} \end{cases}$
(no preferred direction)

* e.) Scaling (for $r \to 0$) $\Rightarrow$ critical: scale elimination, and passing

$$\underline{r},\ \underline{V},\ t \to \lambda \underline{r},\ \lambda^h \underline{V},\ \lambda^{1-h} t$$

i.e. $\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V} = -\underline{\nabla}\, p$

$\underline{V} \to \lambda^a \underline{V}$
$t \to \lambda^b t$

$$\frac{\lambda^a \partial \underline{V}}{\lambda^b \partial t} + \frac{\lambda^{2a}}{\lambda} \underline{V}\cdot\underline{\nabla}\,\underline{V} = -\underline{\nabla}\, p$$

$\hookrightarrow$ from $\underline{\nabla}\cdot\underline{V} = 0$

$\therefore$ $\lambda^{2a-1} = \lambda^{a-b}$

$b = 1-a$

$\Rightarrow \lambda^{-(a-1)} = \lambda^b$

$\therefore$ scalings $\underline{r} \to \lambda \underline{r}$, $\underline{V} \to \lambda^h \underline{V}$, $t \to \lambda^{1-h} t$

Now, turbulence onset $\Rightarrow$ symmetry breaking!

i.e. ① KH : <u>shear</u> breaks $\left\{\begin{array}{l}\text{translational}\\ \text{rotational}\end{array}\right.$ invariance

②

rigid body boundary flow, etc

③ flushing toilet $\left\{\begin{array}{l}\text{space}\\ \text{time}\end{array}\right.$

etc.

However, fully developed turbulence tends to restore symmetry, except near boundarys, on small scale.

b.c.'s $\longleftrightarrow$ boundary conditions

i.e. if $\delta v(\underline{r}, \ell) = \underline{v}(r+\underline{\ell}) - v(\underline{r})$

$\Rightarrow$

$$\delta v(\underline{r}+\underline{\rho}) = \delta v(\underline{r})$$

similarly: isotropy, parity ....

(facilitates scaling approach)

$\underline{H2}$ For $Re \to \infty$ turbulence, at small scales and away from boundaries, the flow is self-similar at small scales

i.e. possesses a unique scaling exponent h s/t

$$\delta v(r, \lambda \underline{\ell}) \to \lambda^h \delta \underline{v}(r, \underline{\ell})$$

($\to$ addresses 2/3 Law)

$\underline{H3}$ With assumptions similar to $H1$, the turbulent flow has a finite, nonvanishing mean rate of dissipation ϵ per unit mass.

$Re \to \infty \Rightarrow \nu \to 0$ with $V_0 = V_{rms}$, ℓ_0 fixed

$$\epsilon = V_0^3 / \ell_0$$

Alternative (not necessary): Kolmogorov's 'Second' Universality Assumption: In the limit of infinite Reynolds number all the small-scale statistical properties are uniquely and universally determined by the scale ℓ and the mean energy dissipation rate ϵ.
(First: $-\frac{4}{5}\epsilon\ell = \langle u^3 \rangle$)

i.e. Implications:

- $\langle \delta v(\ell)^2 \rangle = S_2$?

$S_2 \sim L^2/T^2$, dimensionally

Now $\epsilon \sim L^2/T^3$

$\Rightarrow \langle \delta v(\ell)^2 \rangle \sim \epsilon^{2/3} \ell^{2/3}$ $\Rightarrow$ recovers 2/3 Law.

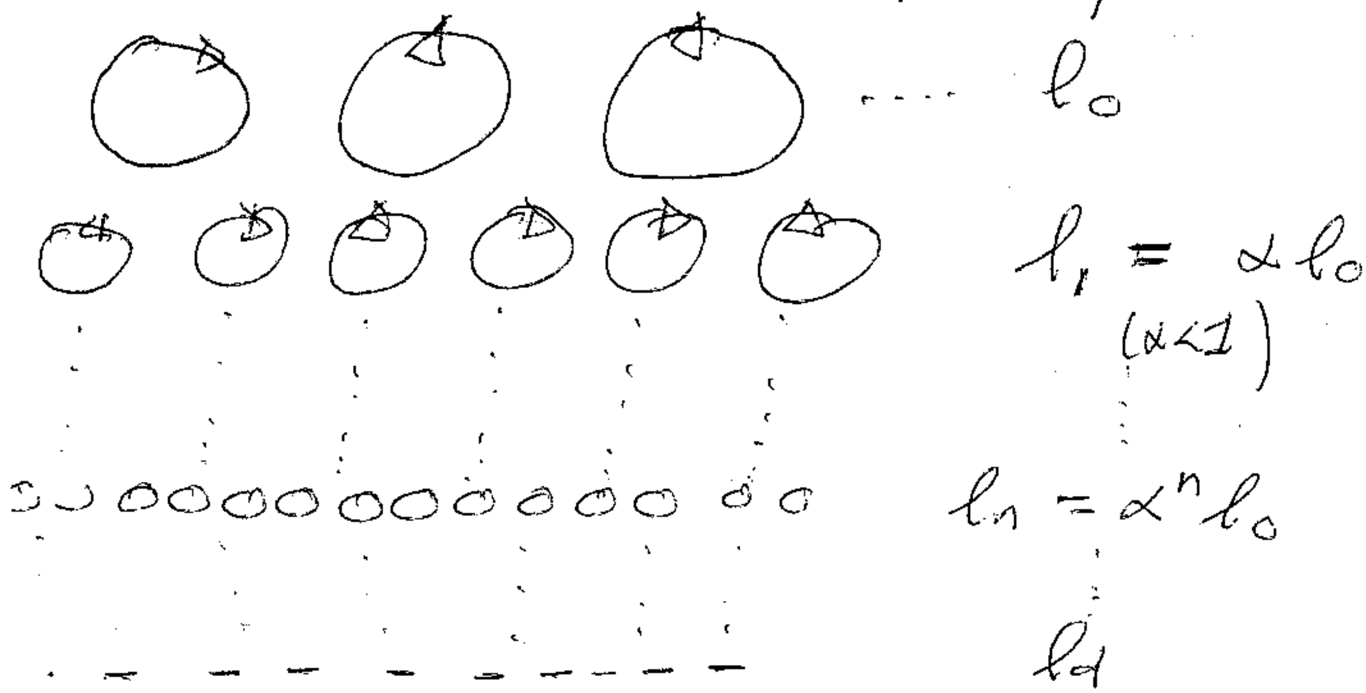
also, implies $h = 1/3 \Leftrightarrow$ scaling exponent, etc.

H1, H2, H3 (+2nd Universality Assumption) $\Rightarrow$

K41 phenomenology.

K41 Phenomenology

Picture : (Richardson) Cascade / Eddy Mitosis



Key Idea : ① → Flux of energy in 'scale space', from l_0 (integral scale) to l_d, (dissipation scale)

② → energy flux is self-similar

ⓐ symmetry restoration.

Flux ↔ ⓑ energy dissipation → finite limit as $\nu \to 0$ (i.e. endpoint re-adjustment)

self-similarity ↔ 2/3 Law $\quad S_2 = C\,(l/l_0)^{2/3}$

$l_0 \to \alpha l_0$ etc. , $C \to C\alpha^2$

Ingredients in K41 Phenomenology:

→ ℓ : scale paramer : eddy scale

→ $v(\ell)$: $\tilde{v}(\ell) \sim \left(\langle \delta v_{\parallel}(\ell)^2 \rangle\right)^{1/2}$

eddy velocity

$$\delta v_{\parallel} \sim \left(\underline{v}(\underline{r}+\underline{\ell}) - \underline{v}(\underline{r})\right) \cdot \frac{\underline{\ell}}{\ell}$$

≡ longitudinal velocity increment

→ V_0 : rms velocity fluctuation (large scale dominates)

$$v(\ell_0) \sim V_0$$

→ $\tau(\ell)$: eddy lifetime / turn-over rate

characteristic rate of transfer thru scale ℓ.

self-similarity : energy thru-put rate is scale invariant

scale ℓ energy

⇒
$$\epsilon = v(\ell)^2 / \tau(\ell)$$

dissipation rate

↳ scale ℓ life-time

energy balance / thru-put eqn!

, ow, $\tau(\ell)$?

Compare $\tau(\ell)$ with $\tau_{ew}(k)$

$\tau(\ell) \to$ 'lifetime' of structure of scale ℓ

$\to$ i.e. time for structure to be distorted out of existence

scales $\ell' \gg \ell$:

$\to$ advect eddy $\to$ apply Galilean boost, but don't affect life-time.

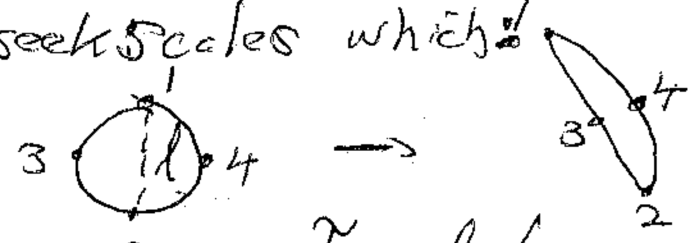
irrelevant! $\to$ symmetry under random Galilean transformations

$\to$ would also violate symmetry restoration.

scales $\ell' \ll \ell$:

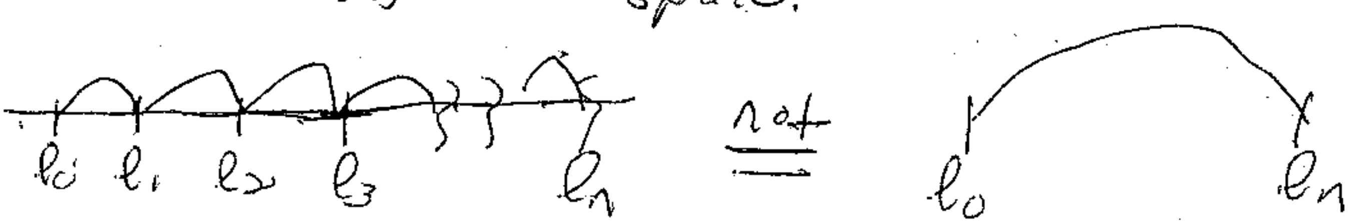
$\to$ irrelevant as very little energy/shear in such eddys/scales

$\therefore$ seek scales which:



$$\tau_\ell \sim \ell / |v_1 - v_2|$$

$$\Rightarrow \tau(\ell) \sim \frac{\ell}{v(\ell)}$$: cascade local in scale space.



$$\epsilon = \frac{v(\ell)^3}{\ell}$$

$$\Rightarrow v(\ell) \sim (\epsilon \ell)^{1/3} \quad : \text{K41 scaling relation}$$

$$\boxed{V(\ell)^2 \sim \epsilon^{2/3} \ell^{2/3}}$$

- "verifies" 2/3 Law
- for spectrum:

if $E(k) = \quad |\tilde{v}(k)|^2$

s/t $E = \int dk\, E(k)$ $\left\{ \begin{array}{l} \text{i.e. absorbs} \\ \text{density of states} \end{array} \right.$

then $V(\ell) = \int_{k_{\ell_{n-1}}}^{k_{\ell_n}} dk\, E(k)$

$$V(\ell)^2 \sim \epsilon^{2/3} \ell^{2/3} = \epsilon^{2/3} k_\ell^{-2/3}$$

$$\Rightarrow \boxed{E(k) = \epsilon^{2/3} k^{-5/3}} \quad : \quad \underline{\text{Kolmogorov Spectrum}}$$

at ℓ_0:

$$V_0 \sim \epsilon^{1/3} \ell_0 \quad \Rightarrow \quad \frac{V_0^3}{\ell_0} = \epsilon .$$

For dissipation scale:

l_d occurs in l-space when cascade terminates
i.e. viscosity asserts itself $\Rightarrow Re(l) \to 1$

$$1/\tau(l) \sim 1/\tau_d = \nu/l^2$$

$$\Rightarrow \quad \epsilon^{1/3} l^{-2/3} = \frac{\nu}{l^2}$$

$$l^{4/3} = \nu/\epsilon^{1/3} \quad \Rightarrow \boxed{l_d = \nu^{3/4}/\epsilon^{1/4}}$$

$\boxed{l_d \equiv \eta \text{, in Frisch}}$

Recall: $\quad \epsilon = \nu\langle(\nabla v)^2\rangle$

$\Rightarrow \nu \to 0 \Rightarrow \langle(\nabla v)^2\rangle$ divergent

$$\langle(\nabla v)^2\rangle = \int_{k_0}^{k_d} dk\, k^2 \epsilon^{2/3} k^{-5/3}$$

$$= \int_{k_0}^{k_d} dk\, k^{1/3} \epsilon^{2/3}$$

$$= k_d^{4/3} \epsilon^{2/3}$$

$$= \frac{\epsilon^{1/3}}{\nu} \epsilon^{2/3} = \epsilon/\nu$$

$\Rightarrow \langle(\nabla v)^2\rangle$ divergent as $\nu \to 0$.

→ Counting Degrees of Freedom

How big is the inertial range:

$$\underset{\#\, l's}{\underset{\Uparrow}{n}} \sim \frac{l_0}{l_d} \sim \frac{l_0}{(\nu^3/\epsilon)^{1/4}}$$

$$\sim \frac{l_0}{\nu^{3/4}}\left(V_0^3/l_0\right)^{1/4} \sim \left(\frac{V_0 l_0}{\nu}\right)^{3/4}$$

$$\sim Re^{3/4}$$

∴ number of degrees of freedom for 3D turbulence is;

$$\boxed{N \sim Re^{9/4}}$$

: would be (minimum) # grid points to resolve range of scales in numerical simulation

Now, i.e. atmospheric boundary layer:

$l_0 \sim 1\,Km$

$l_d \sim 1\,mm$

$$n \sim 10^6 \Rightarrow \begin{cases} N \sim 10^{18} \\ Re \sim 10^8 - 10^9 \end{cases} !$$

⇒ subgrid scale modelling ...

N.B.: Sometimes able to exploit reduced degrees of freedom models, i.e. when some class of scales slaved to others.

Exercises:

⇒ Consider passive scalar, with concentration c:

$$\frac{\partial c}{\partial t} + \underline{v}\cdot\underline{\nabla}c - D\nabla^2 c = \tilde{f}_c$$

c dissipation rate in K41 turbulence $\equiv \alpha$

i.e. $\alpha = \tilde{c}_0^2 \frac{v_0}{l_0}$

⇒ a.) Calculate K41 spectrum for c.

b.) Discuss what if $D \ll \nu$, $D \gg \nu$?

⇒ Consider incompressible turbulence with $M \equiv \frac{v_0}{c_s} \ll 1$.

Show: $\frac{l_d}{l_{mfp}} \sim M^{-1} Re^{1/4}$

⇒ validity of continuum hydrodynamics gets better at high Re.

Particle Separation / Richardson Law

Consider 2 particles (test) in K41 turbulence.
Rate of separation?

→ larger eddys advect both

→ smaller eddys do ~ nothing

⇒ divergence controlled by eddys of scale $\ell \sim |\underline{x}_1 - x_2|$.

∴ if $\lambda \equiv |\underline{x}_1 - \underline{x}_2|$

$$\Rightarrow \quad \frac{d\lambda}{dt} = v(\lambda) = \epsilon^{1/3}\lambda^{1/3}$$

$$\lambda^{2/3} = \epsilon^{1/3} t$$

$$\therefore \quad \boxed{\lambda \sim \epsilon^{1/2} t^{3/2}}$$

Richardson's 3/2 Law

N.B.: Non-diffusive!

$$\lambda^2 \sim \epsilon t^3 \Rightarrow \boxed{\tau_{sep.} \sim \lambda^{2/3} / \epsilon^{1/3}}$$

N.B.:

→ process is self-accelerating ⇒ large eddys move faster

- non-diffusive.

→ [Momentum] Flux Driven Turbulence

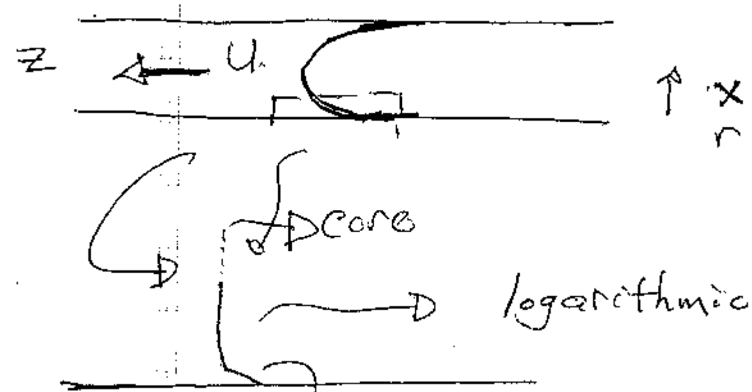
Turbulent Pipe Flow

(cf. Landau, Lifshitz "Fluid Mechanics")

Till now → homogeneous flow in a periodic box
→ cascade in scale space (Kolmogorov)

Now → inhomogeneous flow in a pipe
→ momentum transport in a turbulent boundry layer (Prandtl)

Consider turbulent pipe flow:



→ core

→ logarithmic profile (inertial sublayer) (Virtually all high Re experiments → "profile consistency")

→ linear profile (viscous sublayer)

Common features of pipe flow:

- linear → logarithmic $u(x)$ profile
- logarithmic profile persists over a broad range of Re

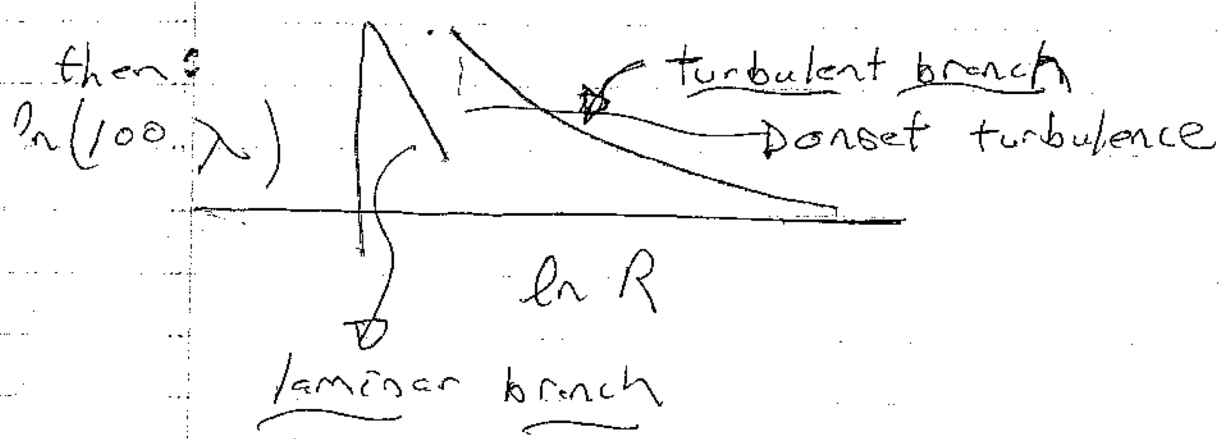
$$\left(Re = 2Ua/\nu \right)$$

∴ logarithmic profile "universal" (Prandtl "Law of the Wall")

- resistance ~~discontinuously~~ increases with increasing Re,

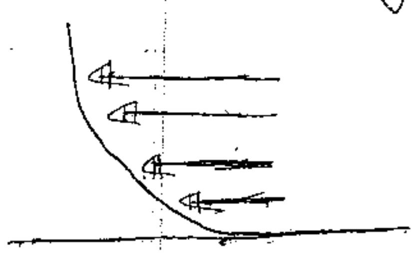
$$\lambda = \frac{2a\,\underbrace{\Delta p/\ell}_{\text{pressure drop/length}}}{\underbrace{\tfrac{1}{2}\rho U^2}_{\text{mean flow energy}}}$$

then:



- turbulent resistance curve universal.

∴ What is going on?



no slip boundary condition

$$U = U(x) \underset{x \to 0}{\longrightarrow} 0$$

$$\therefore\ U = U(x) \Rightarrow \begin{cases} \text{momentum flux} \\ \text{to wall!} \end{cases}$$

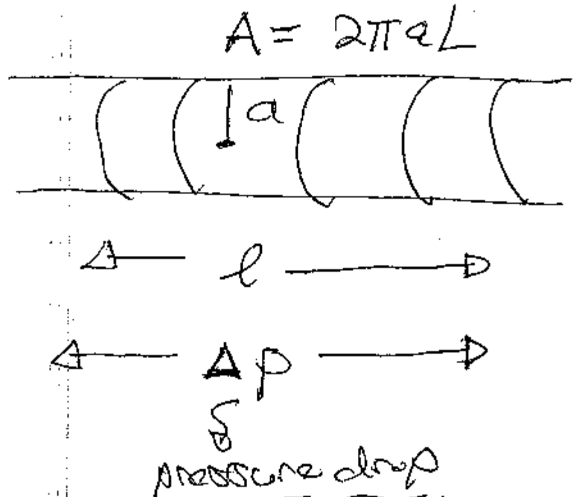
→ Momentum flux to wall ⇒ stress on the wall

→ wall stress must balance pressure drop, for steady flow

so wall stress : ρU_*^2
$U_* \equiv$ friction velocity

i.e.
$$\underbrace{\underline{v}\cdot\nabla\underline{v} - \nu\nabla^2\underline{v}}_{F \sim SA} = \underbrace{-\nabla p}_{F\sim pA} \sim \frac{\Delta p}{\ell}$$
$$2\pi V_*^2 \ell R \sim \Delta P \pi R^2$$

$\therefore \quad \rho U_*^2 2\pi a \ell = \Delta p \pi a^2$



Force on wall ≃ $\rho U_*^2 A_{wall}$

(Pressure Drop) A_{Flow} = Force on Fluid

ition only ⇒ $\rho U_*^2 (2\pi a \ell) = (\Delta p)\pi a^2$

$$U_* = \left[(\Delta p/2\rho)\left(\frac{a}{2\ell}\right)\right]^{1/2}$$

Friction Velocity

u_* ≡ friction velocity
≡ "typical" velocity of turbulence in turbulent pipe

Deriving the inertial sublayer profile:

i) dimensional reasoning

in pipe flow inertial sublayer, have

3 dimensional parameters ρ, σ, x

$\underbrace{\rho}_{\text{density}}$, $\underbrace{\sigma}_{\text{wall stress } u_*}$, $x \to$ distance from wall

Key Point: ~~Assumption~~ Assumption of scale invariance

on scale $l_{vs} = \frac{\nu}{u_*} < x < a$

→ universality of logarithmic profile motivates scale invariance assumption

now, seek velocity gradient du/dx,

$\frac{du}{dx}$: u_*, x, ρ

so simplest form for dU/dx is:

$$\frac{dU}{dx} = \frac{U_*}{X}$$

$$\Rightarrow \boxed{\begin{aligned} U &= \frac{U_*}{K} \ln(X/X_0) \\ &= \frac{U_*}{K} \ln X + \text{const.} \end{aligned}}$$

→ logarithmic profile (consequence of scale invariance in pipe flow)

→ $K \sim .4$ universal constant → Von-Karman constant

→ $X_0 \Leftrightarrow$ width of viscous sublayer $\sim \nu/U_*$

ii.) Physical Reasoning

stationary flow ⇒

momentum flux to wall = pressure drop

$$\therefore \quad \underbrace{\langle \tilde{v}_x \tilde{v}_z \rangle}_{\text{Reynolds stress}} = u_*^2$$

$$\rho \langle \tilde{v}_x \tilde{v}_z \rangle = \underbrace{\Gamma_p}_{\text{momentum flux}}$$

$$\Gamma_p / \rho = u_*^2$$

Now, to calculate $\langle \tilde{v}_x \tilde{v}_z \rangle$:

→ take velocity fluctuation as generated by <u>mixing</u> of $U(x)$, so

$$\Rightarrow \quad \tilde{v}_z \sim \underbrace{\ell}_{\text{"mixing length"}} \frac{\partial U}{\partial x}$$

analogous to Chapman - Enskog expansion, i.e.

$\ell \Leftrightarrow \ell_{mfp}$.

$\tilde{v}_x \Leftrightarrow v_{Th}$.

here scale invariance $\Leftrightarrow$ $\ell \sim x$

mixing length set by distance from wall

so

$$\langle \tilde{v}_x \tilde{v}_z \rangle = \langle V_x \ell \rangle \frac{\partial U}{\partial x}$$

$$\cong U_* x \frac{\partial U}{\partial x}$$

$\nu_T = U_* x$ $\rightarrow$ "eddy viscosity" "turbulent viscosity" $\Big\rangle$ Key concept

$\Rightarrow$ rate of turbulent transport of momentum

then momentum balance $\Rightarrow$

$$U_* x \frac{\partial U}{\partial x} = U_*^2$$

$\Rightarrow$

$$U = \frac{U_*}{\kappa} \ln(x/x_0)$$

$\rightarrow$ Logarithmic Profile

$\rightarrow$ Law of the Wall

Some comments:

→ as in K41, clear phenomenology critical to guiding the approximations ↔ scale invariance

→ "Mixing length theory always works .... provided you know the mixing length..."
— P.D.

→ why a single value of velocity, i.e. u_*?

Consistent with mixing length hypothesis, velocity fluctuations generated by mixing of mean flow gradient, i.e.

⇒

$$\tilde{V} \sim l \frac{\partial u}{\partial x} \sim x \frac{\partial u}{\partial x}$$

$$\sim \cancel{x} \frac{u_*}{\cancel{x}}$$

absence of preferred scale.

consistent. ∴ Assumption consistent with:
- logarithmic profile
- scale invariance.

⇒ viscous sublayer / cut-off of inertial layer ?

∴ when: $\nu_T < \nu$ { molecular viscosity dominates mixing

⇒ $u_* x \lesssim \nu$

$$x \lesssim \nu / u_* \equiv x_0$$

viscous sublayer scale.

In viscous sublayer, flow linear:

$$\nu \frac{\partial u}{\partial x} = u_*^2$$

$$\therefore \quad u = \frac{u_*^2}{\nu} x$$

⇒ note effect of turbulence is to:

- flatten profile – { higher transport at fixed wall stress
- reduce central velocity
- limit Q (quality factor)

– matching, for const:

$$x_0 = \nu / u_* \quad 80$$

$$u = \frac{u_*}{\kappa} \ln\left(\frac{u_* y}{\nu}\right)$$

Note: Flow in viscous sublayer is turbulent, but mixing there effected by dissipation range scales ⇒ linear profile

Now – turbulent dissipation ?.

Consider NSE:

$$\frac{\partial \hat{\underline{V}}}{\partial t} + \hat{\underline{V}} \cdot \nabla \hat{\underline{V}} + \langle V_z \rangle \frac{\partial}{\partial z} \hat{\underline{V}} + \hat{V}_x \frac{\partial}{\partial x} \langle V_z \rangle$$

$$= -\nabla \hat{p} + \nu \nabla^2 \hat{\underline{V}}$$

$\hat{\underline{V}} \cdot$ and avg ⇒

$$\frac{\partial}{\partial t} \langle \hat{V}^2 \rangle + \langle \hat{\underline{V}} \cdot \hat{\underline{V}} \cdot \nabla \hat{\underline{V}} \rangle + \langle V_z \rangle \cancel{\langle \hat{\underline{V}} \cdot \frac{\partial}{\partial z} \hat{\underline{V}} \rangle}$$

odd

$$+ \langle \hat{V}_x \hat{V}_z \rangle \frac{\partial}{\partial x} \langle V_z \rangle = -\cancel{\langle \hat{\underline{V}} \cdot \nabla \hat{p} \rangle} - \nu \langle (\nabla \hat{V})^2 \rangle$$

i.b.p.

obviously: $\nu \underbrace{\langle (\nabla \tilde{v})^2 \rangle}_{\text{small scale dissipation}} = \underbrace{\nu_T \left(\frac{\partial u}{\partial x}\right)^2}_{\text{input} \to \text{mean flow mixing}}$

and

$$\epsilon = (u_* x) \left(\frac{u_*}{x}\right)^2 \qquad \text{(ignoring } k\text{)}$$

$$= \frac{u_*^3}{x}$$

→ sets dissipation rate.

i.e. $\epsilon = \frac{V_0^3}{\ell}$ $\qquad V_0 \leftrightarrow u_*$, $\ell \leftrightarrow x$

→ ϵ finite as $\nu \to 0$ (i.e. viscous sublayer gradient diverges then)

Additional References:

- S.B. Pope, "Turbulent Flows"

- H. Tennekes and J. Lumley, "A First Course in Turbulence"

For net energy budget:

$$\partial_t \mathcal{E} = -\langle \hat{V}_x \hat{V}_z \rangle \frac{\partial \langle V_z \rangle}{\partial x} - \nu \langle (\nabla \tilde{V})^2 \rangle$$

input to fluctuations by relaxation of mean shear flow (Reynolds work)

dissipation of fluctuation energy by viscosity

∴ Can define:

$$\epsilon = \langle \hat{V}_x \hat{V}_z \rangle \frac{\partial U}{\partial x}$$

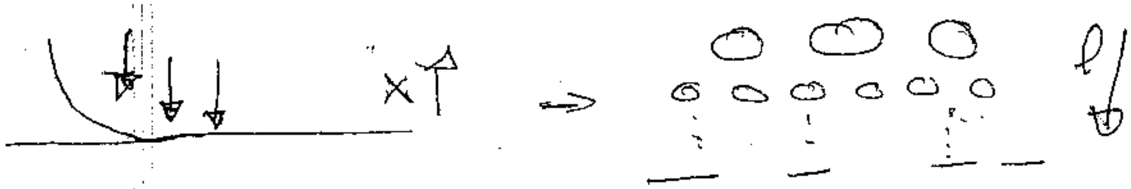
turbulent dissipation rate

and using mixing length theory:

$$\langle \hat{V}_x \hat{V}_z \rangle = U_* x \frac{\partial U}{\partial x}$$

$$\Rightarrow \quad \epsilon = (U_* x)\left(\frac{\partial U}{\partial x}\right)^2 = \nu_T \left(\frac{\partial U}{\partial x}\right)^2$$

rate of "heating" by turbulent relaxation of mean flow.



→ Now, interesting to tabulate comparison between Pipe Flow and K41 Problem

Pipe Flow (Prandtl)	K41 (Kolmogorov)
scales: a, x, ν/u_* invariance: $x \to$ real space	ℓ_0, ℓ_n, ℓ_d $\ell \to$ scale space
inertial sublayer viscous sublayer	inertial range dissipation range
alance: $u_*^2 = \nu_T \dfrac{\partial u}{\partial x}$	$\epsilon = \dfrac{v(\ell)^2}{\tau(\ell)}$
ynamics: eddy viscosity $\nu_T = u_* x$	turn-over rate $1/\tau(\ell) = \dfrac{v(\ell)}{\ell}$
sult: $u = \dfrac{u_*}{\kappa} \ln(x)$	$v(\ell) = \epsilon^{1/3} \ell^{1/3}$
universal profile	universal spectral scaling
issipation: $\nu = \nu_T$ $x_0 = \nu/u_*$	$v(\ell)/\ell = \nu/\ell^2$ $\ell_d = \nu^{3/4}/\epsilon^{1/4}$.

→ Practical Issues

Resistance Law ⇔ Pipe Flows.

have: $\frac{\nu}{u_*} < x \leq a$ ↓ radius

can push to $x \sim a$, with logarithmic accuracy

$$U \cong \frac{u_*}{k} \ln \left(\frac{u_* a}{\nu} \right)$$

but
$$V_* = u_* = \left(\frac{a}{\ell} \frac{\Delta P}{2\rho} \right)^{1/2}$$

⇒ can re-write:

$$U = \left(\frac{a \Delta P}{2\rho \ell k^2} \right)^{1/2} \ln \left(a \left(\frac{a \Delta P}{2\rho \ell} \right)^{1/2} / \nu \right)$$

Convenient to define:

$$\lambda = \frac{2a \overset{\to \text{drive}}{\Delta P} / \ell}{\underbrace{\tfrac{1}{2} \rho U^2}_{\text{flow KE}}}$$

→ friction factor / resistance coefficient

$\Rightarrow$ taking $R_e = 2aU/\nu$

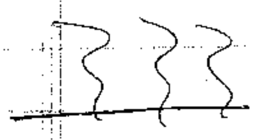
can re-write friction law as:

$$\begin{cases} 1/\sqrt{\lambda} = .88 \ln\left(R_e\sqrt{\lambda}\right) - \underbrace{.80}_{\text{phenom.}} \\ R_e = 2aU/\nu \\ \lambda = \dfrac{2a\,\Delta p/\ell}{1/2\,\rho U^2} \end{cases}$$

- good fit to pipe flow data.

Problems :

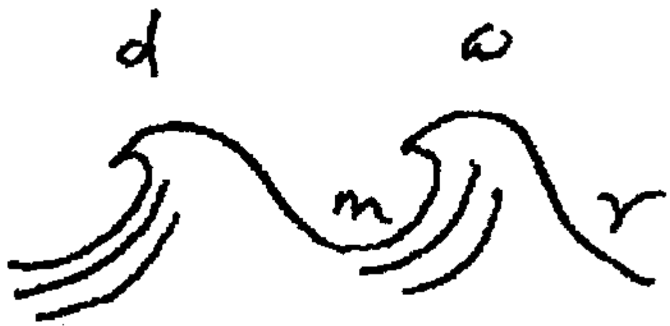
1 a.) A very strong explosion, with energy released ΔE, creates a spherical blast wave in an atmosphere of Pressure P, density ρ. Use dimensional analysis to derive the radius of the blast front as a function of time, i.e. $r(t)$? When does this scaling fail?

b.)  A hot surface produces thermal convection above it. Assuming the convection is turbulent, use scaling arguments to calculate the temperature profile above the plate assuming the hot plate drives a surface heat flux Q. (See Chapter 5, Landau).

Why Wave "Kinetics"?

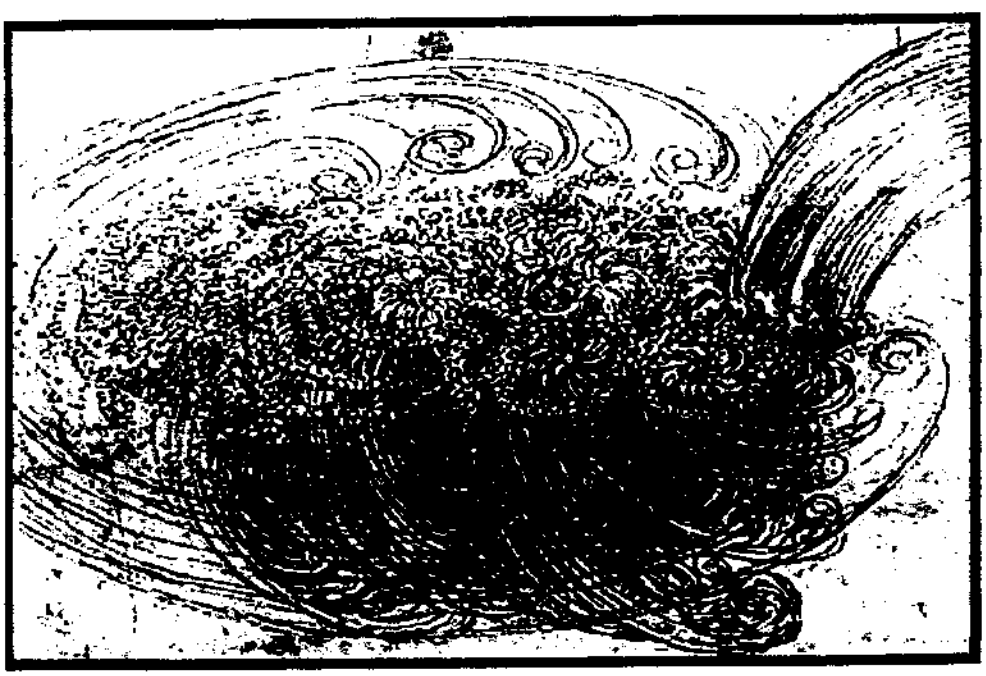
" A wave is never found alone, but is mingled with as many other waves as there are uneven places in the object where the said wave is produced. At one and the same time there will be moving over the greatest wave of a sea innumerable other waves proceeding in different directions"

- Leonardo da Vinci
  Codice Atlantico, c.1500.



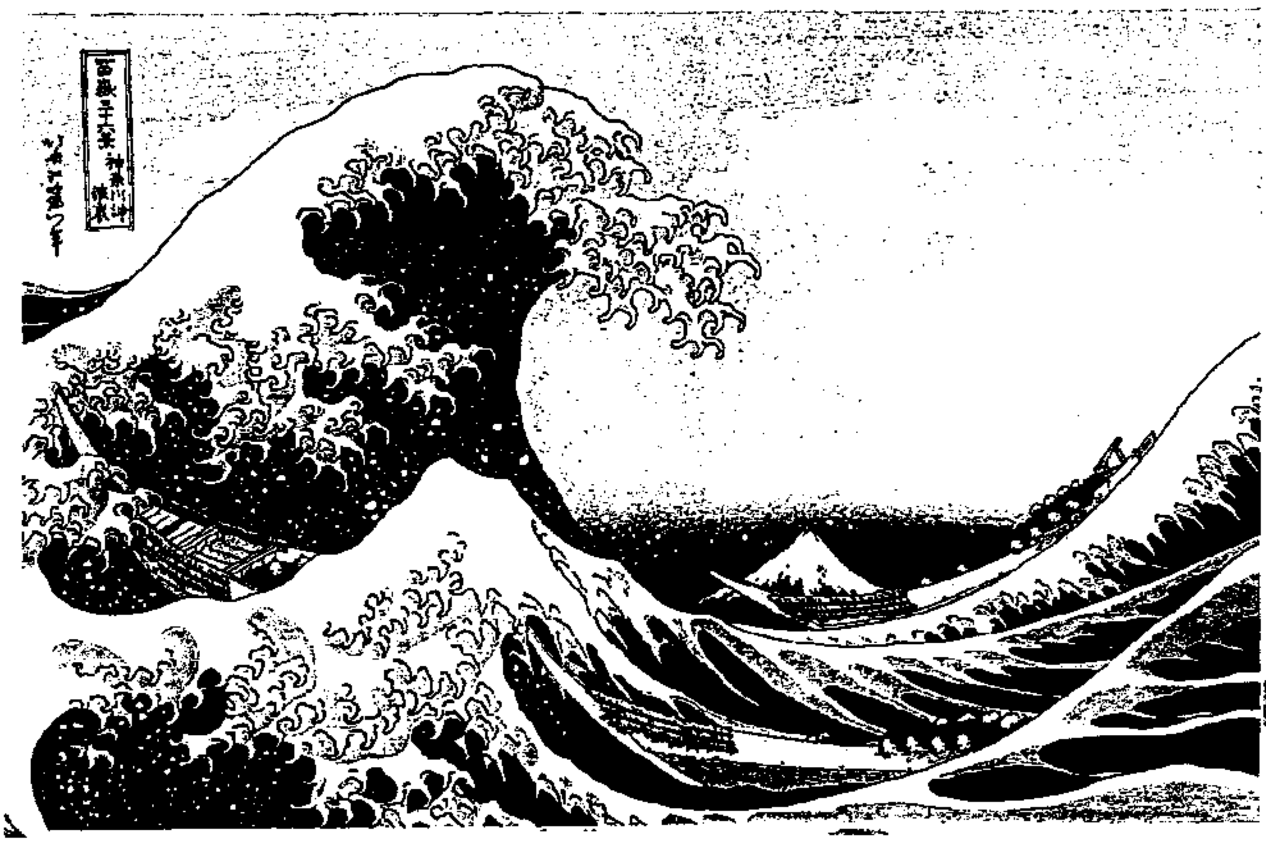
abnollilamioqul opodr m
qmr allops rinlbuynqn
dn msd.qn.m ior

[Leonardo
on waves ...]



From Asian art.....

The great wave at Kanagawa Hokusai



Outline

i.) Introduction and Basic Ideas

→ what is wave kinetics?

→ basic structure and limitations?!

ii.) Why Bother? - Utility of Wave Kinetics

→ particle ⊕ quasi-particle picture

applications:

- energetics → conservation laws
- "radiation hydrodynamics" → wave forces
- structure formation
  - → Langmuir turbulence
  - → zonal flow formation
- non-locality → turbulence spreading

iii.) <u>Basic Theory</u>

→ intuitive arguments

→ outline of Whitham derivation (see supplement I)

→ full wave kinetics

→ special items: drift waves, 2D fluids, etc.

iv.) Applications I → Conservation Relations

→ Energetics: simple and not-so-simple . . .

→ Parallel Momentum

v.) Applications II → Structure Formation (<u>Introduction</u>)

→ Langmuir Turbulence (see supplement II)

→ Zonal Flow Formation

a) Introduction and Basic Ideas

→ Wave Kinetics or Quasi-Particle Picture

- Boltzmann-like equation for wave population

$$\frac{dN}{dt} = C(N) \; ; \quad N(\underline{k}, \underline{x}, t) \equiv \left\{ \begin{array}{l} \text{population} \\ \text{density} \end{array} \right.$$

$$\underbrace{\frac{\partial N}{\partial t} + (\underline{V}_{gr} + \underline{V}) \cdot \underline{\nabla} N - \frac{\partial}{\partial \underline{x}} (\omega + \underline{k} \cdot \underline{V}) \cdot \underline{\nabla}_k N}_{LHS}$$

$$= \underbrace{C(N)}_{RHS} \qquad \text{a la' Boltzmann}$$

LHS: $\left. \frac{dN}{dt} \right|_{rays}$

characteristics

$$\left. \begin{array}{l} \dfrac{dx}{dt} = \underline{V}_{gr} + \underline{V} \\[2ex] \dfrac{dk}{dt} = -\dfrac{\partial}{\partial \underline{x}} (\omega + \underline{k} \cdot \underline{V}) \end{array} \right\} \begin{array}{l} \text{eikonal} \\ \text{equations} \\ \Rightarrow \text{ray trajectories} \\ \text{for waves} \end{array}$$

for LHS (cont'd): short λ, high ω wave ⇄ long λ low ω modulator

eikonal theory ⟷ <u>scale separation</u>

$\tilde{\omega}$, $\tilde{V}$ → modulation field with scales → $\left\{ \begin{array}{l} \underline{q} \to \text{wave vector} \\ \Omega \to \text{frequency} \end{array} \right.$

∴ LHS ↓ disparate scale interaction $\left\{ \begin{array}{l} \omega_{\underline{k}} > \Omega, \ \underline{q} \cdot \underline{V}, \ |\underline{V}|; \text{etc.} \\ |\underline{k}| > |\underline{q}|; \text{etc.} \end{array} \right.$

<u>if</u> $C(N) \to 0$ and $\omega > \Omega$, etc
→ adiabatic invariance....

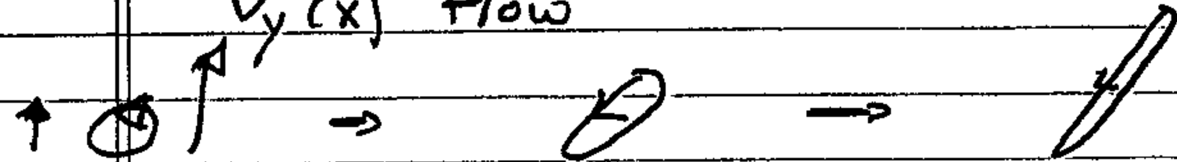
$\frac{dN}{dt} = 0$ ⟹ $N \equiv$ wave action density $= \mathcal{E}_{\underline{k}} / \omega_{\underline{k}}$

$\mathcal{E}_{\underline{k}}$ ↓ energy density

$\frac{dN}{dt} = 0$ ↓ { wave population density conserved along rays.

⟹ tells how wave population responds to modulation field.

i.e. classic example → shearing (coherent) $V_y(x)$ flow



k_x increases (refraction) $\Rightarrow \dfrac{dk_x}{dt} = -\dfrac{\partial}{\partial x}\left(\omega + V_y(x) k_y\right)$

$$\cong -k_y \frac{\partial V_y(x)}{\partial x}$$

how does energy change?

$N = \mathcal{E}/\omega_k$, $V_y' < \omega_k$

$\Rightarrow N \cong N_0$ const
adiabatic invariant

$\Rightarrow$

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}(N\omega_k) = N_0 \frac{\partial \omega}{\partial k} \cdot \frac{dk}{dt}$$

$$= N_0 v_{grx} \frac{dk_x}{dt}$$

$$= N_0 v_{grx} \left(-k_y \frac{\partial V_y(x)}{\partial x}\right)$$

$\dfrac{d\mathcal{E}}{dt} \gtrless 0 \rightarrow \mathrm{sgn}: -v_{gr} k_y V_y'$

$$\sim \frac{k_y^2 k_x V_y' \rho_s^2 V_*}{(1+k_\perp^2\rho_s^2)^2}$$

stochastic → zonal flows (later)

RHS $\rightarrow$ non Action-conserving wave interactions

i.e.

$$C(N) = \sum_{\substack{\underline{k}', \underline{k}'' \\ \oplus = k}} \left\{ C_1(\underline{k}', \underline{k}'') N_{\underline{k}'} N_{\underline{k}''} - C_2(\underline{k}, \underline{k}') N_{\underline{k}'} N_{\underline{k}} \right\} * \delta(\omega_{\underline{k}''} - \omega_{\underline{k}} - \omega_{\underline{k}'})$$

effective collision operator (non-adiabatic wave interactions) $\rightarrow$ {topic for another lecture}

so obvious analogy:

$$\left\{ \frac{\partial N}{\partial t} + (\underline{v}_{gr} + \underline{V}) \cdot \underline{\nabla} N - \frac{\partial}{\partial \underline{X}}(\omega + \underline{k} \cdot \underline{V}) \cdot \nabla_k N = C(N) \right.$$

LHS $\rightarrow$ N conserving

C(N): wave-wave interaction

$\updownarrow$

$$\left\{ \frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla} f + \frac{q}{m} \underline{E} \cdot \frac{\partial f}{\partial \underline{v}} = C(f) \right.$$

LHS $\Rightarrow$ f conserving

C(f): particle collisions

$\therefore$ $\boxed{N = \text{quasi-particle density}}$

so: Wave kinetics:

→ Boltzmann-like equation for wave population density (action density → usually) N

→ LHS → wave-mean interactions (slow)

RHS → wave-wave interaction (break adiabatic invariance)

→ useful when: scale separation between mean, waves

→ useful because: can apply tricks from particle kinetics to wave population.

i.e. if $C(N) = -\nu_{eff}(N - \langle N \rangle)$

how calculate flux of wave momentum if $\langle N \rangle = \langle N(x) \rangle$?
(radiation pressure)

⇒ "radiation hydrodynamics":

see: Mihalas and Mihalas
Zeldovich and Raizer

→ useful because: natural way to formulate conservation laws of quasi-linear theory

→ useful because: natural way to formulate structure formation →

- Langmuir turbulence
- zonal flow formation

Supplement I:

Variational Theory of Wave Kinetics

(Whitham)

→ Where does $N \Leftrightarrow$ Action emerge from?

→ adiabatic invariance ⇔ conservation law

Noether's Thm? → symmetry!

Phase symmetry of wave train underlies action conservation and wave kinetics

→ derivation of wave kinetics via variational principle

→ Variational Theory (Whitham)

Now, consider system, like ideal MHD, which can be described in terms of a displacement $\underline{\xi}$ such that

$$\underline{\xi} = \mathrm{re}\left\{A e^{i\phi} + A^{*} e^{-i\phi}\right\}$$

then relevant wave equation can be derived from:

$$\delta S = \delta \int dt \int d\underline{x} \; \mathcal{L}(\underline{\xi}) = 0$$

↳ Lagrangian density

Now, if write Lagrangian density in terms phase ϕ and amplitude a, have:

$$S = \int dt \int d\underline{x} \; \mathcal{L}(-\phi_t, \phi_{\underline{x}}, a)$$

where $\omega = -\phi_t = \partial\phi/\partial t$

$$\underline{k} = \phi_{\underline{x}} = \nabla\phi$$

→ this neglects all corrections to eikonal theory (WKB) i.e. all corrections to $\underline{k}$, ω, amplitude, etc.

→ L, above, corresponds to period averaged Lagrangian — ϕ indeterminate to constant

$$\Rightarrow \quad \delta S \equiv \delta \int dt \int dx \, \mathcal{L}(-\phi_t, \phi_x, a)$$

so have $\underline{\underline{2}}$ variational equations:

1) $\delta S / \delta a = 0$

2) $\delta S / \delta \phi = 0$

Now, within scope of linear theory

$$\mathcal{L} = G(\omega, k)\, a^2$$

i.e. for MHD, can write:

$$\mathcal{L} = \frac{1}{2} \rho \, \dot{\underline{\xi}}^2 - \frac{1}{2} \rho \left[D(\underline{k}, \underline{x}, t) \right]^2 \underline{\xi}^2$$

→ eikonal form of potential energy (from stiffness matrix)

and if $\underline{\xi} = \underline{A} e^{+i\phi} + \underline{A}^* e^{-i\phi}$

$$\mathcal{L} = \frac{1}{2} \rho \left(\frac{\partial \phi}{\partial t} \right)^2 |A|^2 - \frac{1}{2} \left[D(\nabla \phi, \underline{x}, t) \right]^2 |A|^2$$

is concrete form of avg. Lagrangian.

$\Rightarrow \quad G(\omega, k) = \frac{1}{2}\rho\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left[\Omega(\nabla\phi, \underline{x}, t)\right]\right]|A|^2$

Now, 1) $\Rightarrow \quad \frac{\partial S}{\partial a} = 0$

$$\Rightarrow \boxed{G(\omega, k) = 0}$$

but: $G = \omega^2 - \left[\Omega(\underline{k}, \underline{x}, t)\right]^2 = 0$

is just dispersion relation!

2) $\Rightarrow \quad \delta S/\delta\phi = 0$

so

$$\delta S = \int dt \int dx \left\{ \frac{\partial \mathcal{L}}{\partial(-\phi_t)} \delta(-\phi_t) + \frac{\partial \mathcal{L}}{\partial(\phi_{\underline{x}})} \delta\phi_{\underline{x}} \right\}$$

$$= \int dt \int dx \left\{ \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(-\phi_t)}\right) - \frac{\partial}{\partial \underline{x}} \cdot \left(\frac{\partial \mathcal{L}}{\partial \phi_x}\right) \right\} \delta\phi$$

$\Rightarrow \quad \delta S = 0 \quad \Rightarrow$

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \omega}\right) - \underline{\nabla} \cdot \left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0}$$

so have: $\mathcal{L} = G(\omega, \underline{k})\, a^2$

$$\Rightarrow \quad G(\omega, \underline{k}) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \omega}\right) - \nabla\cdot\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

Now $\quad G(\omega, \underline{k}) = 0$

$$\Rightarrow \quad \frac{\partial G}{\partial \omega} d\omega + \frac{\partial G}{\partial \underline{k}} d\underline{k} = 0$$

$$\therefore \quad \underline{V}_{gr} = \frac{d\omega}{d\underline{k}} = -\frac{\partial G/\partial \underline{k}}{\partial G/\partial \omega}$$

$\left(\begin{array}{l} \text{i.e. } G(k, \omega) = 0 \\ dG = 0 = \frac{\partial G}{\partial \omega} d\omega + \frac{\partial G}{\partial \underline{k}} \cdot d\underline{k} \end{array}\right)$

$$\Rightarrow \quad \frac{\partial}{\partial t}\Big(\underbrace{\frac{\partial G}{\partial \omega} a^2}_{\frac{\partial \mathcal{L}}{\partial \omega}}\Big) + \underline{\nabla}\cdot\Big[-\frac{\partial G/\partial \underline{k}}{\partial G/\partial \omega} \underbrace{\frac{\partial G}{\partial \omega} a^2}_{\frac{\partial \mathcal{L}}{\partial \underline{k}}}\Big] = 0$$

and so $\quad N \equiv \frac{\partial G}{\partial \omega} a^2$

$$\frac{\partial N}{\partial t} + \nabla\cdot\left(\underline{V}_{gr} N\right) = 0$$

(N not yet action ....)

- Now, can further note for $\boxed{G \text{ invariant to time trans.}}$

$\Rightarrow$ energy is conserved.

so, $\exists$ energy conservation equation (from Noether's Thm $\Leftrightarrow$ symmetry).

Now, note have:

can proceed by working with:

$$\partial \mathcal{L}/\partial a = 0 \quad\rightarrow\quad G(\omega, \underline{k}) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \omega}\right) - \underline{\nabla}\cdot\left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

$$\frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}} \qquad \left(\frac{\partial}{\partial t}\frac{\partial \phi}{\partial \underline{x}} = \frac{\partial}{\partial \underline{x}}\frac{\partial \phi}{\partial t}\right)$$

$$\underline{\nabla}\times \underline{k} = 0 \qquad (\underline{k} = \underline{\nabla}\phi)$$

$\Rightarrow$ have $$\frac{\partial}{\partial t}\left(\omega \mathcal{L}_\omega - \mathcal{L}\right) + \nabla\cdot\left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

check: $$\underbrace{\omega \frac{\partial \mathcal{L}_\omega}{\partial t}}_{\overset{?}{=} 0} - \frac{\partial \mathcal{L}}{\partial t} + \underline{\nabla}\cdot\left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}}\right) \quad + \frac{\partial \mathcal{L}}{\partial \omega}\frac{\partial \omega}{\partial t}$$

$$= \frac{\partial \mathcal{L}}{\partial \omega}\frac{\partial \omega}{\partial t} + \omega \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right) - \nabla \cdot \left(\omega \frac{\partial \mathcal{L}}{\partial \underline{k}}\right) - \frac{\partial \mathcal{L}}{\partial t}$$

$$= \cancel{\omega \underline{\nabla} \cdot \left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right)} - \cancel{\omega \underline{\nabla} \cdot \left(\frac{\partial \mathcal{L}}{\partial \underline{k}}\right)} - \frac{\partial \mathcal{L}}{\partial \underline{k}} \cdot \underline{\nabla} \omega - \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial \omega}\frac{\partial \omega}{\partial t}$$

$$= + \frac{\partial \mathcal{L}}{\partial \underline{k}} \cdot \frac{\partial \underline{k}}{\partial t} + \frac{\partial \mathcal{L}}{\partial \omega}\frac{\partial \omega}{\partial t} - \frac{\partial \mathcal{L}}{\partial t}$$

$$= 0 \checkmark$$

constructed form of energy eqn:

$$\frac{\partial}{\partial t}\left(\omega \frac{\partial \mathcal{L}}{\partial \omega} - \mathcal{L}\right) + \underline{\nabla} \cdot \left(-\omega \frac{\partial \mathcal{L}}{\partial \underline{k}}\right) = 0$$

but $\mathcal{L} = 0$, so energy density of wave is

$(G(\omega, \underline{k}) = 0)$

$$\boxed{\mathcal{E} = \omega \frac{\partial \mathcal{L}}{\partial \omega}}$$

above states energy conservation

$$\Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \omega} = \frac{\mathcal{E}}{\omega} \equiv \text{Action density!}}$$

$\Rightarrow$ <u>now</u> N is action density!

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial \sigma}{\partial \omega} a^2\right) + \underline{\nabla}\cdot\left[\frac{-\partial\sigma/\partial \underline{k}}{\partial\sigma/\partial\omega}\frac{\partial\sigma}{\partial\omega} a^2\right] = 0}$$

$$\Rightarrow \quad \boxed{\frac{\partial N}{\partial t} + \underline{\nabla}\cdot\left[\underline{v}_{gr} N\right] = 0}$$

$$\omega N = \mathcal{E}$$

Now note if write Vlasov-like equation:

$$\frac{\partial N}{\partial t} + \underline{v}_{gr}\cdot\underline{\nabla} N - \frac{\partial \omega}{\partial \underline{x}}\cdot\frac{\partial N}{\partial \underline{k}} = 0$$

Liouville $\Rightarrow$

$$\frac{\partial N(\underline{k},\underline{x},t)}{\partial t} + \underline{\nabla}\cdot\left[\underline{v}_{gr} N\right] + \frac{\partial}{\partial \underline{k}}\cdot\left[-\frac{\partial\omega}{\partial\underline{x}} N\right] = 0$$

and $\int d\underline{k}$, with assumption of narrow spread in $\underline{k}$
for N

$\Rightarrow$

$$\frac{\partial}{\partial t} N(\underline{x},t) + \underline{\nabla}\cdot\left[\underline{v}_{gr} N\right] = 0$$

Recall :

→ Hamiltonian structure of eikonal theory, etc. ⇒

$$\frac{\partial \rho(\underline{k},\underline{x},t)}{\partial t} + \underline{v}_{gr}\cdot\underline{\nabla}\,\rho(\underline{k},\underline{x},t) - \frac{\partial\omega}{\partial\underline{x}}\cdot\underline{\nabla}_k\,\rho(\underline{k},\underline{x},t) = 0$$

→ physical arguments suggest $\rho = \frac{\mathcal{E}}{\omega} = N$ wave action density

→ Variational Approach

$$S = \int dt \int d^3x\, \mathcal{L} \quad , \qquad \mathcal{L} = G(\omega,\underline{k})\, a^2$$

$$\omega = -\partial\phi/\partial t = -\phi_t$$

$$\underline{k} = \underline{\nabla}\phi = \phi_{\underline{x}}$$

$$\delta S = 0$$

but two parameters varied $\left\langle \begin{array}{l} a \\ \phi \end{array}\right.$

$\delta S/\delta a = 0 \Rightarrow G(\omega,\underline{k}) = 0$ → dispersion relation

$$\delta S/\delta\phi = 0 \Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\omega}\right) - \frac{\partial}{\partial\underline{x}}\cdot\left(\frac{\partial\mathcal{L}}{\partial\underline{k}}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial G}{\partial\omega}a^2\right) - \frac{\partial}{\partial\underline{x}}\cdot\left(\frac{\partial G}{\partial\underline{k}}a^2\right) = 0$$

and time translation symmetry and $G = 0$ ⇒

$$\mathcal{E} = \omega \frac{\partial \epsilon}{\partial \omega} a^2 \Rightarrow N = \frac{\mathcal{E}}{\omega} = \frac{\partial \epsilon}{\partial \omega} a^2$$

and $\frac{\partial \epsilon}{\partial \underline{k}} a^2 = \underline{V}_{gr} \cdot N$

→ Helpful Reminder:

recall, for electrostatic plasma waves

if $\epsilon(\omega, \underline{k}) = 0 \Rightarrow$ dispersion relation

then $$\mathcal{E}_k = \frac{\partial}{\partial \omega}(\omega \epsilon)\Big|_{\omega_{\underline{k}}} \frac{|E_k|^2}{8\pi}$$

$$= \omega_k \frac{\partial \epsilon}{\partial \omega}\Big|_{\omega_k} \frac{|E_k|^2}{8\pi}$$ → wave energy density

$$\therefore N_k = \frac{\partial \epsilon}{\partial \omega}\Big|_{\omega_k} \frac{|E_k|^2}{8\pi}$$

and $$\underline{\Gamma}_k = -\frac{\partial \epsilon}{\partial \underline{k}}\Big|_{\omega_k} \frac{|E_k|^2}{8\pi}$$ → wave energy density flux

$$= \underline{V}_{gr} N_{\underline{k}}$$

since $G(\underline{k}, \omega) = 0$, so along rays

$$dG = d\omega \frac{\partial G}{\partial \omega} + d\underline{k} \cdot \frac{\partial G}{\partial \underline{k}} = 0$$

$$d\omega/d\underline{k} = -(\partial G/\partial \underline{k})/(\partial G/\partial \omega)$$

etc.

we have Vlasov-like eqn. in $\underline{x}$, $\underline{k}$ phase space

$$\frac{\partial N}{\partial t} + \underline{v}_g \cdot \frac{\partial N}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

and continuity-type eqn. in $\underline{x}$ space:

$$\frac{\partial N}{\partial t} + \underline{\nabla} \cdot \left[\underline{v}_{gr} N \right] = 0$$

Observe:

→ order of derivatives matters, but Liouville helps

→ continuity-type eqn. for packets

→ useful to note that total derivative of $\underline{k}$, following packet

$$\frac{d\underline{k}}{dt} = \frac{\partial \underline{k}}{\partial t} + \underline{v}_{gr} \cdot \frac{\partial \underline{k}}{\partial \underline{x}}$$

$$= -\left(\frac{\partial \omega}{\partial \underline{x}}\right) + \underline{v}_{gr} \cdot \frac{\partial \underline{k}}{\partial \underline{x}}$$

if $\omega = \Omega(\underline{k}, \underline{x}, t)$ from $\epsilon = 0$

$$= -\frac{\partial \Omega}{\partial \underline{k}} \cdot \frac{\partial \underline{k}}{\partial \underline{x}} - \frac{\partial \Omega}{\partial \underline{x}} + \underline{v}_{gr} \cdot \frac{\partial \underline{k}}{\partial \underline{x}} = -\frac{\partial \Omega}{\partial \underline{x}}$$

$$= -\left(\partial \omega / \partial x\right)_{\underline{k}}$$

no conflict with $\frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}}$

Langmuir Turbulence and Disparate Scale Interaction

Contents:

1.) Overview

*2.) Langmuir Turbulence — { Basics; ~~Eikonal Analysis~~ }

3.) Adiabatic Response vs. Adiabatic Elimination (Zwanzig–Mori)

*4.) Langmuir Collapse → Self Focusing

i.) Overview — Non-local Energy Transfer

the problem of two interacting mode/structure populations is generic

Generic structure } "P_{eff}"

High Frequency Small Scale

Low Frequency Larger Scale

"strain field"

effective stress, pressure of high on low

⇒ (Reynolds) stress

slow variation of high frequency wave < parameter, envelope

⇒ MODULATION

→ partial list of examples ⇒ {structure formation and maintenance

*① Langmuir Turbulence — CLASSIC EXAMPLE

high frequency → electron plasma wave
low frequency → ion-acoustic wave

radiation pressure

interactions: h→L : plasmon pressure / ponderomotive force
L→h : refraction $dk/dt = -\frac{\partial \omega}{\partial x}$

② Disparate Scale IW Interaction — Induced Diffusion

high frequency → internal wave
low frequency → long wave length inertial / IW's, "currents"

h→L : Wave Reynolds stresses, transport
$\langle \tilde{V}_z \tilde{V}_H \rangle$, $\langle \tilde{V}_z \tilde{\rho} \rangle$

L→H : random (i.e. hence 'diffusion') refraction
$\langle \delta k^2 \rangle$ in RPA theory

③ Drift Wave — Convective Cell / Zonal Flow
N.B. $\omega_{Low} \to 0$
high frequency ⇒ drift wave — HM
low frequency → convective cell
$\partial_t \nabla^2 \phi + \nabla\phi \times \hat{z} \cdot \nabla \nabla^2 \phi + \gamma_d \nabla^2 \phi = 0$
$k_{y\,cell} \to 0$ ⇒ zonal flow

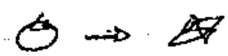
$h \to \ell$: drift wave Reynolds stress — stochastic
i.e. for zonal flow, $\langle \tilde{V}_x \tilde{V}_y \rangle$ radiation stress

parametric modulational — coherent
instability

$L \to H$: induced diffusion in k_r $\Rightarrow$ "random shearing"

$$\frac{dk_r}{dt} = -\frac{\partial}{\partial x}(\phi + k_\theta \tilde{V}_E)$$

$$k_r = k_r^{(0)} - k_\theta V_E' t, \quad \langle \delta k_r^2 \rangle = k_\theta^2 \langle \tilde{V}_E'^2 \rangle \tau_c$$



④ Acoustic Wave + Vortex (i.e. flow-induced noise)

high frequency $\to$ sound wave
low frequency $\to$ vortex

$h \to L$: ponderomotive acoustic stresses, Pressure

$L \to H$: refraction of acoustic wave ray in vertical
flow field.

and (not really) last and by no means least...

(5) turbulent dynamo problem, i.e. $\langle \underline{B} \rangle$ from $\langle \tilde{\underline{V}} \times \tilde{\underline{B}} \rangle$

'high frequency' → fluid turbulence / waves

'low frequency' → mean $\langle B \rangle$ field

question: is, in some sense, a pre-existing bath/gas of turbulence (MHD) "unstable" to amplification of $\langle B \rangle$.

high freq → low freq: mean EMF $\langle \tilde{V} \times \tilde{\underline{B}} \rangle$, calculated ≈ perturbatively

$l \to h$: evolving concept
i.e. $\langle B \rangle \to \langle \tilde{B}^2 \rangle$ → back reaction on EMF
$\langle \underline{B} \rangle$ → bending-induced stabilization of drive

N.B. 'Classic dynamo' is 'generic fluid turbulence' driven but can have wave-driven dynamo.

Note: (1) → (5) have very similar theoretical structure

Here, will focus on Langmuir turbulence.

ii.) Basics of Langmuir Turbulence

→ Zakharov Equations

Basic Physics: – acoustic wave density perturbation refracts plasma waves
– plasma waves form pressure field for acoustic wave

plasma wave: $\underline{E} = \underbrace{\mathcal{E}(\underline{x},t)}_{\substack{\text{envelope} \\ \text{(slow)}}} \, \underline{e}_0 \exp[\underbrace{i(\underline{k}\cdot\underline{x} - \omega t)}_{\text{carrier}}]$

$$\omega^2 = \omega_{pe}^2 + \gamma_T k^2 v_{th}^2$$

$$\therefore \quad -\frac{\partial^2 \underline{E}}{\partial t^2} = \omega_p^2 (1 + \underbrace{\delta n}_{\text{refracting perturbation}}) \underline{E} - \gamma_T v_T^2 \nabla^2 \underline{E}$$

$$\Rightarrow \quad \omega_p^2 \mathcal{E} + i\omega_p \frac{\partial \mathcal{E}}{\partial t} - \underbrace{\cancel{\frac{\partial^2 \mathcal{E}}{\partial t^2}}}_{h.o.} = \omega_p^2 \mathcal{E} + \delta n \, \omega_{p0}^2 \mathcal{E} - \gamma_T v_T^2 \nabla^2 \mathcal{E}$$

$$\therefore \quad \boxed{\frac{i}{\omega_p} \frac{\partial \mathcal{E}}{\partial t} + \underbrace{\lambda_D^2 \nabla^2 \mathcal{E}}_{\text{diffraction}} = \underbrace{\delta n \, \mathcal{E}}_{\text{refraction}}}$$

For acoustic wave:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla P \qquad P = P_{Thermal} + P_{\substack{\text{plasma}\\ \text{wave}}}$$

$$= P_{Th} + \frac{\partial}{\partial \omega}(\omega \epsilon)\Big|_{\omega_{p0}} \frac{|E|^2}{8\pi}$$

→ energy density of plasma wave

→ also via 'ponderomotive force' approach

$\therefore$

$$\hat{P} = \underbrace{C_s^2 \frac{\hat{\rho}}{\rho_0}}_{\text{thermal}} + \underbrace{\frac{|\tilde{E}|^2}{4\pi \rho_0}}_{\text{radiation}}$$

$$\frac{\hat{\rho}}{\rho_0} = \delta n$$

$$\Rightarrow \boxed{\left(\frac{\partial^2}{\partial t^2} - C_s^2 \nabla^2\right) \delta n = \frac{\nabla^2 |\tilde{E}|^2}{4\pi n_0 m_i}}$$

acoustic wave eqn.

Further, let: $\omega_{pe} t \rightarrow t$

$\mathbf{x}/\lambda_D \rightarrow \mathbf{x}$

$|E|^2/4\pi \rightarrow |\tilde{E}|^2/4\pi n_0 T_0$

yield dimensionless Zakharov Equations:

envelope Eqns.

$$\boxed{\begin{aligned} &\left(i\frac{\partial}{\partial t} + \nabla^2\right)\mathcal{E} - \delta n\,\mathcal{E} = 0 \\ &\left(\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2\right)\delta n - c_s^2\nabla^2|\mathcal{E}|^2 = 0 \end{aligned}}$$

$\cancel{c_s^2 = m_e/m_i}$

- coupled envelope equations for

  plasma wave envelope: $\mathcal{E}(\underline{x}, t)$
  density perturbation: δn

- (1) with $\delta n \to 0 \Rightarrow$ Schroedinger eqn. for free particle
  (2) with $|\mathcal{E}|^2 \to 0 \Rightarrow$ acoustic wave equation

- space-time scale separation is crucial to both derivation.

- can estimate c/s' dimensional analysis (in acoustic equation)

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2\right) \delta n = + c_s^2 \nabla^2 |\mathcal{E}|^2$$

$$\underbrace{\frac{\partial^2}{\partial t^2}}_{O(1/T^2)} \qquad \underbrace{c_s^2\nabla^2}_{O(c_s^2/L^2)}$$

$\therefore$ $\Rightarrow$ $1/T^2 \ll c_s^2/L^2$ $\Rightarrow$ subsonic/adiabatic limit

$\Rightarrow$ $1/T^2 \gg c_s^2/L^2$ $\Rightarrow$ supersonic/non-adiabatic limit

subsonic case: $\left(\cancel{\frac{\partial^2}{\partial t^2}} - c_s^2 \nabla^2\right) \delta n = c_s^2 \nabla^2 |\mathcal{E}|^2$

$$\delta n \cong -|\mathcal{E}|^2 \qquad \text{"cavity"}$$

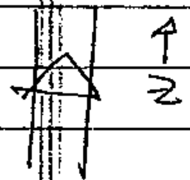
$$\Rightarrow \boxed{\left(i\frac{\partial}{\partial t} + \nabla^2\right)\mathcal{E} + |\mathcal{E}|^2\mathcal{E} = 0}$$

NLS equation
corresponds to adiabatic Zakharov eqn.

N.B. in MHD, similar procedure for Alfvenic carrier yields DNLS.

supersonic eqn. limit: can't avoid two equations.

– subsonic Langmuir turbulence problem maps to optical self-focusing



light beam

$$\underbrace{n^2}_{\text{index}} = \Big(1 + \underbrace{\frac{\Delta n}{n}\frac{|E|^2}{E_0^2}}_{\substack{\text{intensity dependent} \\ \text{perturbation in refractive index}}}\Big)$$

dependence

$$\nabla^2 \underline{E} + \frac{\omega^2 n^2}{c^2}\underline{E} = 0$$

$$\underline{E} = \mathcal{E}(z, \underline{x}_\perp)\, e^{ikz} e^{-i\omega t} \qquad \omega^2 = c^2 k^2$$

$\Rightarrow$

$$\boxed{2ik\frac{\partial \mathcal{E}}{\partial z} + \nabla_\perp^2 \mathcal{E} + k^2 \frac{\Delta n}{n} |\mathcal{E}|^2 \mathcal{E} = 0}$$

– NLS!

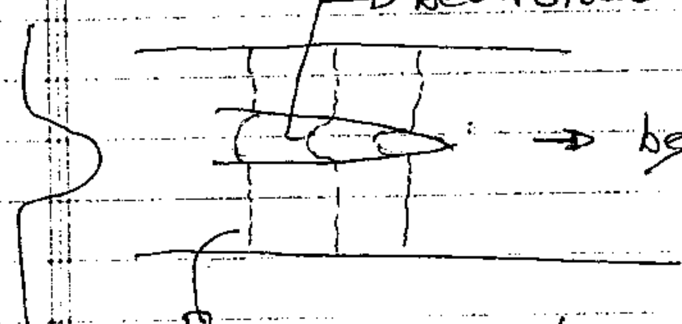
Physics of self-focusing:

$$V_{phase}^2 = c^2/n^2 = c^2 \Big/ 1 + \frac{\Delta n}{n}|\mathcal{E}|^2$$

$\therefore$ phase speed lower in regions of high intensity i.e. consider progression of phase fronts in beam of finite width (rays $\perp$ phase fronts)

n.b. phase front = iso-phase surface

$\rightarrow$ beam slower in regions of peak intensity



$\rightarrow$ beam focuses.

phasefronts $\rightarrow$ slower here

focus $\Leftrightarrow$ singularity formation!

N.B. Aside - Langmuir turbulence of interest both as disparate scale interaction problem and as route to and understandable example of singularity formation in finite time

- recall finite time singularity is essence of turbulence problem

more on this in collapse discussion

ϵ independent ν in 3D NST $\Rightarrow$

$\langle |\nabla \underline{v}|^2 \rangle \sim \epsilon/\nu$

singular velocity gradient

- Linear Theory of $\left\{ \begin{array}{l} \text{NLS} \\ \text{Self-Focusing} \end{array} \right.$

$$i\,\partial_t \mathcal{E} + \nabla^2 \mathcal{E} + |\mathcal{E}|^2 \mathcal{E} = 0$$

$$\mathcal{E} = \underbrace{A}_{\text{amplitude}} e^{i\phi \to \text{phase}}$$

- 2 fields $\to$ NLS is complex equation!
- similar to treatment of cycles in CGL

$\Rightarrow$

$$\frac{\partial \phi}{\partial t} + (\nabla\phi)^2 - \left(\frac{\nabla^2 A}{A} + |A|^2 \right) = 0$$

$$\frac{1}{A}\frac{\partial A}{\partial t} + \nabla^2 \phi + 2\frac{\nabla\phi \cdot \nabla A}{A} = 0$$

so linearizing:

$$A_0 \frac{\partial \tilde{\phi}}{\partial t} + \nabla^2 \tilde{A} + |A_0|^2 \tilde{A} = 0$$

$$\frac{\partial \tilde{A}}{\partial t} + A_0 \nabla^2 \tilde{\phi} = 0 \qquad \left\{ \begin{array}{c} A \\ \phi \end{array} \right\} = F_{q,\Omega}\, e^{i(\underline{q}\cdot\underline{x} - \Omega t)}$$

$$\Rightarrow \quad \Omega^2 = -q^2\left(|A_0|^2 - q^2\right)$$

$$\gamma_q^2 = q^2\left(|A_0|^2 - q^2\right)$$

self focusing (destabilizing)

diffraction (spreading $\Rightarrow$ stabilizing)

NLS tends to amplify $|\tilde{\mathcal{E}}|^2$, δn

$\Rightarrow$ instability for $\frac{|E_0|^2}{4\pi n T} > k^2 \lambda_D^2$

self-attraction

diffraction

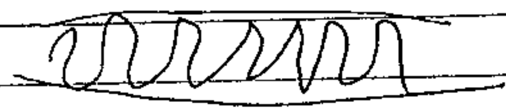
$\Rightarrow$ linear marginality at:

$$k^2 \sim \frac{1}{\lambda_D^2} |\mathcal{E}_0|^2 / 4\pi n_0 T_0$$

$\Rightarrow$ nonlinear evolution / ultimate fate ?

→ Langmuir Turbulence ?

- Zakharov equations, etc. is envelope analysis
- assumes narrow band envelope on carrier



- what of turbulence → multiple structures.

Now, can write:

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2\right)\delta n = c_s^2 \nabla^2 \frac{|E|^2}{4\pi n_0 T}$$

but - exploit scale separation to note adiabatic invariant

- invariant is wave action density of plasma waves

$$N = \mathcal{E}_k / \omega_k \qquad \mathcal{E}_{\omega_k} = \frac{\partial}{\partial \omega}(\omega \epsilon)\Big|_{\omega_k} \frac{|E_k|^2}{8\pi}$$

$$\omega_k^2 = \omega_p^2 (1 + \gamma_e k^2 v_T^2)$$

– view $N(\underline{k}, \underline{x}, t)$ as population density of waves/excitons

N.B. Interesting questions occur re: just what is N?? → what is "the correct" N??

① i.e. for DWT,

$$N = \mathcal{E}/(\omega_*/1+k_\perp^2\rho_s^2)$$

action density

$$= (1+k_\perp^2\rho_s^2)^2 |e\phi_k/T|^2 / \omega_*$$

$$= \underbrace{\Omega}_{\text{potential enstrophy}} / \omega_*$$

$\left\{ \frac{d\omega}{dt}\Big|_{\beta t} = 0 \right.$ is H–M equation

action density ≡ $\#_{\text{density}}$ of waves

potential enstrophy ≡ $\#_{\text{density}}$ of rotons/vortices

for $dk_y/dt = 0$ (zonal flow shear field i.e. $\underline{V} = V_y(x)\,\hat{y}$)

$N = \Omega$ up to constant factor ⇒ ambiguity dissolved (but not resolved)

What of $\frac{d}{dt}k_y \neq 0$?? → technical. see Smolyakov, P.D. PoP 2000

② related: are "waves" necessary for wave kinetic picture

→ No! see B. Dubrulle and S. Nazarenko, Physica D 1997

i.e. considered evolution of 2D turbulence in 2D large scale slow strain field

in 2D, $\frac{d}{dt}\omega = 0$ $\quad\quad \omega = \nabla^2\phi$

→ wavelet field (windowed F.T.)

showed: $N = |\omega(k, x, t)|^2$ → enstrophy

$$\frac{dN}{dt} = \frac{\partial N}{\partial t} + \underline{V}\cdot\nabla N - \frac{\partial}{\partial \underline{x}}(\underline{k}\cdot\underline{V})\cdot\frac{\partial N}{\partial \underline{k}} = 0$$

$\underline{V} \equiv$ strain field.

key: N tied to intensity of quantity conserved along particle trajectories

Suggests in 2D MHD, "magnetic" $N = |A(k, x, t)|^2$.

wavelet field of magnetic potential

→ investigate

→ spectral transfer?

Now, can write:

$$P_{\substack{ES\\ \text{wave}}} = \omega_{p_0} N \quad , \text{ so}$$

$$\frac{\partial \tilde{\rho}}{\partial t} = -\rho_0 \underline{\nabla} \cdot \underline{\tilde{V}}$$

$$\rho_0 \frac{\partial \underline{\tilde{V}}}{\partial t} = -C_s^2 \underline{\nabla} \tilde{\rho} - \underline{\nabla} \int dk \; \omega_k \tilde{N}(\underline{k}, \underline{x}, t)$$

① { acoustic wave with } { radiation pressure }

of:

$$\frac{d}{dt} N + (\underline{V}_{gr} + \underline{V}) \cdot \underline{\nabla} N - \frac{\partial}{\partial \underline{x}} (\omega + \underline{k} \cdot \underline{V}) \cdot \underline{\nabla}_k N = 0$$

$$\Rightarrow \frac{\partial \tilde{N}}{\partial t} + V_{gr} \cdot \underline{\nabla} \tilde{N} = + \frac{\partial}{\partial \underline{x}} \tilde{\Omega} \cdot \frac{\partial \langle N \rangle}{\partial \underline{k}}$$

② plasma wave population density

where took $\langle N \rangle = \langle N(\underline{k}) \rangle$ homogeneous
($\Rightarrow$ what is $\langle N \rangle$?, what sets it?)

so

→ ①, ② together "close loop" of disparate scale feed back.

→ N eqn is effectively a "subgrid scale" model for effect of plasma waves (unresolved) on resolved ion-acoustic wave

reduced degree of freedom $\left\{ \tilde{k}, \tilde{x} \leftrightarrow \tilde{N} \right.$ → somewhat related to k-ε model idea

perturbations of plasma wave field

→ what happens?

$$N = \frac{\varepsilon}{\omega} = N_0, \text{ const.}$$

(# plasmons conserved)

$$\varepsilon = N\omega$$

$$\frac{d\varepsilon}{dt} = N \frac{\partial \omega}{\partial k} \cdot \frac{dk}{dt}$$

$$= N\, \underline{v}_{gr} \left(-\frac{\partial}{\partial \underline{x}} \omega_p\, \delta n \right)$$

For multiple modulation field (i.e. Langmuir turbulence) ⇒

$\delta n \Rightarrow \delta \tilde{n}$

$N \to \langle N \rangle + \tilde{N}$

induced by $\delta \tilde{n}$

so $\frac{d\mathcal{E}}{dt} = \left\langle (\langle N \rangle + \tilde{N}) \frac{\partial \omega}{\partial \underline{k}} \cdot \left(-\frac{\partial}{\partial \underline{x}} \omega_{p_0} \delta n \right) \right\rangle$

$= -\frac{\partial \omega}{\partial \underline{k}} \cdot \left\langle \frac{\partial}{\partial \underline{x}} \omega_{p_0} \delta n \, \tilde{N} \right\rangle$

need calculate this correlator.

Proceed a la' QLT:

$\rangle = \sum_{\underline{q}, \Omega} -i \underline{q} \, \omega_{p_0} \delta n_{-\underline{q}, -\Omega} \tilde{N}_{\underline{q}, \Omega}$

$\tilde{N}_{\underline{q}, \Omega} = \frac{\omega_{p_0} i \underline{q} \, \tilde{n}_{\underline{q}, \Omega}}{-i(\Omega - \underline{q} \cdot V_{gr})} \cdot \partial \langle N \rangle / \partial \underline{k}$

$\langle \; \rangle = \sum_{\underline{q}, \Omega} i \underline{q} \omega_{p_0} \delta n_{-\underline{q}, -\Omega} \, \delta n_{\underline{q}, \Omega} \frac{\underline{q} \cdot \partial \langle N \rangle / \partial \underline{k}}{(\Omega - \underline{q} \cdot V_{gr})}$

$= \sum_{\underline{q}, \Omega} \omega_{p_0}^2 \underline{q} \underline{q} \, |\delta n_{\underline{q}, \Omega}|^2 \pi \delta(\Omega - \underline{q} \cdot V_{gr}) \cdot \frac{\partial \langle N \rangle}{\partial \underline{k}}$

so

$$\boxed{\frac{d\mathcal{E}}{dt} = -\underline{V}_{gr}(\underline{k}) \cdot \sum_{\underline{q},\Omega} \omega_{pi}^2 \, \underline{q}\,\underline{q} \, |\delta n_{\underline{q},\Omega}|^2 \pi \delta(\Omega - \underline{q}\cdot\underline{V}_{gr}) \cdot \frac{\partial\langle N\rangle}{\partial \underline{k}}}$$

now:

$$-\ \underline{V}_{gr} = \gamma V_T^2 \underline{k}/\omega_{\underline{k}} > 0 \qquad (\underline{V}_g \parallel \underline{k})$$

so

$$\Rightarrow \frac{d\mathcal{E}}{dt} < 0 \iff \frac{\partial\langle N\rangle}{\partial \underline{k}} > 0$$

but $\frac{d}{dt}\mathcal{E}_{\substack{\text{plasma}\\ \text{wave}}} < 0 \Rightarrow \frac{d}{dt}\mathcal{E}_{\substack{\text{ion}\\ \text{acoustic}\\ \text{wave}}} > 0$

(will prove energy conservation later, but obvious from means of $d\mathcal{E}/dt$)

so growth of ion-acoustic mode at expense of plasmon gas iff

plasmon population inversion!

→ for this type of "inverse cascade".

— What does the calculation mean?

$$\frac{\partial N}{\partial t} + (\underline{v}_g + \underline{V})\cdot\nabla N - \frac{\partial(\omega_p \delta n)}{\partial \underline{x}}\cdot\frac{\partial N}{\partial \underline{k}} = 0$$

$$\frac{\partial\langle N\rangle}{\partial t} - \frac{\partial}{\partial \underline{k}}\cdot\left\langle\left(\frac{\partial}{\partial \underline{x}}\omega_p\delta n\right)\cdot\tilde{N}\right\rangle = 0$$

but $\mathcal{E} = \omega_k N$

$\Rightarrow$ can recognize

$$\int d\underline{k}\;\omega_p\frac{\partial\langle N\rangle}{\partial t} = \int d\underline{k}\;\frac{\partial\langle\mathcal{E}\rangle}{\partial t} = \int\left[-\underline{v}_{gr}(k)\cdot\left\langle\frac{\partial}{\partial \underline{x}}\omega_p\delta n\,\tilde{N}\right\rangle\right]d\underline{k}$$

and
$$\begin{cases}\dfrac{\partial\langle N\rangle}{\partial t} = \dfrac{\partial}{\partial \underline{k}}\cdot\underline{\underline{D}}\cdot\dfrac{\partial\langle N\rangle}{\partial \underline{k}}\\[2ex] \underline{\underline{D}} = \omega_p^2\sum\limits_{q,\Omega}\underline{q}\,\underline{q}\,|\delta n_{q,\Omega}|^2\,\pi\,\delta(\Omega - \underline{q}\cdot\underline{v}_{gr})\end{cases}$$

$\therefore$

induced diffusion of plasma waves by density perturbations (acoustic waves)!

∴ - induced diffusion of plasmons generates plasmon $\underline{k}$-space flux

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial \underline{k}} \cdot \Gamma_{\underline{k}} = 0$$

$$\vec{\Gamma}_{\underline{k}} = -\underline{\underline{D}}_{k} \cdot \frac{\partial \langle N \rangle}{\partial \underline{k}}$$

then $\dfrac{d\mathcal{E}}{dt} = -\underline{V}_{gr} \cdot \underline{\underline{D}} \cdot \dfrac{\partial \langle N \rangle}{\partial \underline{k}}$

∴ flux of certain sign will relax energy.

$$\left(\underline{V}_{gr} \cdot \frac{\partial \langle N \rangle}{\partial \underline{k}} > 0 \right)$$

- what is origin of irreversibility? - What prevents 'back cascade' forward?

→ ray chaos! i.e. overlap of:

$$\Omega = \underline{q} \cdot \underline{V}_{gr}(\underline{k})$$

resonances

- estimate of $D_{induced}$?

$$D \sim \langle q_{r}^{2} \tilde{\delta n}^{2} \rangle \tau_{ac} \,, \quad \tau_{ac} \sim \left(\Delta | \Omega - \underline{q} \cdot V_{gr} | \right)^{-1}$$

→ Conventional Approach : Modulational Instability

→ do gas plasma waves unstable to density / ion-acoustic perturbation ?

→ if so, under what conditions ?

i.e.

$$\frac{\partial \tilde{\rho}}{\partial t} = -\rho_0 \underline{\nabla} \cdot \tilde{\underline{V}}$$

plasmon radiation

$$\rho_0 \frac{\partial \tilde{\underline{V}}}{\partial t} = -\underline{\nabla} \tilde{P} - \underline{\nabla} \int dk\, \omega_{p_0} \tilde{N}(\underline{k}, \underline{x}, t)$$

$$\frac{\partial \tilde{N}}{\partial t} + \underline{v}_{gr} \cdot \underline{\nabla} \tilde{N} = \frac{\partial}{\partial \underline{x}} \omega_{p_0} \frac{\delta \rho}{\rho_0} \cdot \frac{\partial \langle N \rangle}{\partial \underline{k}}$$

acoustic $(\underline{q}, \Omega)$

$$\left(\frac{\delta \rho}{\rho_0} = \frac{\hat{\rho}}{\rho_0} \right)$$

linear crank ⇒

$$\Omega^2 = q^2 c_s^2 - q^2 \frac{\omega_{p_0}^2}{m_i} \int d\underline{k}\, \frac{\underline{q} \cdot \partial \langle N \rangle / \partial \underline{k}}{(\Omega - \underline{q} \cdot \underline{v}_{gr})}$$

usual acoustic term

radiation pressure

proceeding a la' P.T.:

$$\Omega = \Omega^r_q + i\gamma_q$$

$$\Omega^2 = \Omega^{r\,2}_q + 2i\Omega^r_q\gamma_q - \cancel{\gamma_q^2}$$

$$\Omega^2_{r_q} + 2i\Omega_{r_q}\gamma_q = q^2 c_s^2 - \frac{q^2}{m_i}\omega_{pe}^2 \int d\underline{k}\; \underline{q}\cdot\frac{\partial\langle N\rangle}{\partial \underline{k}} \left\{ \frac{P}{\Omega - \underline{q}\cdot \underline{v}_{gr}} - i\pi\,\delta(\Omega - \underline{q}\cdot \underline{v}_{gr}) \right\}$$

so

$$\begin{cases} \Omega^2_{r_q} = q^2 c_s^2 \\ 2i(q c_s)\gamma_q = i\pi \dfrac{q^2}{m_i}\omega_{pe}^2 \displaystyle\int d\underline{k}\; \underline{q}\cdot\frac{\partial\langle N\rangle}{\partial \underline{k}}\,\delta(\Omega - \underline{q}\cdot \underline{v}_{gr}) \end{cases}$$

$\Rightarrow$ instability of ion acoustic wave if $\left.\dfrac{\partial\langle N\rangle}{\partial k}\right|_{\text{reson}} > 0$, as before

$\rightarrow$ analogy to Landau resonance instability obvious.

Can note:

- population inversion needed for modulational instability

$\leftrightarrow$ if $\partial\langle N\rangle/\partial k\Big|_{\text{res}} < 0 \rightarrow$ damping

- resonance is just induced diffusion limit of 3 wave resonance, i.e.

$$|\underline{k}| < |\underline{k}'| \qquad |\omega_{\underline{k}}| < |\omega_{\underline{k}'}|$$

$$\frac{i}{\left[\omega_{\underline{k}+\underline{k}'} - \omega_{\underline{k}} - \omega_{\underline{k}'}\right]} = \frac{i}{\left[\cancel{\omega_{\underline{k}'}} + \underline{k}\cdot\frac{\partial\omega}{\partial\underline{k}'} - \cancel{\omega_{\underline{k}'}} - \omega_{\underline{k}}\right]}$$

$$= i\Big/\left(\underline{k}\cdot\frac{\partial\omega}{\partial\underline{k}'} - \omega_{\underline{k}}\right)$$

in consistent notation

$$= i\Big/\left(\underline{q}\cdot\underline{V}_{gr} - \Omega\right)$$

i.e. $\underbrace{V_{phase}}_{\text{long waves}} \sim \underbrace{V_{gr}}_{\text{short waves}}$

→ Also ask: how does modulational instability saturate?

- quasilinear flattening of N

$$\frac{\partial \langle N \rangle}{\partial t} = \frac{\partial}{\partial k} D_k \frac{\partial \langle N \rangle}{\partial k}$$

if $\left.\frac{\partial \langle N \rangle}{\partial k}\right\}_{\text{reson}} \to 0$ $\quad \gamma_q \to 0$ $\quad$ i.e. source/drive depletion

- exploiting analogy of $\begin{cases} \text{plasma wave} \leftrightarrow \text{"particle"} \\ \text{ion-acoustic wave} \leftrightarrow \text{"wave"} \end{cases}$

can explore:

- trapping of ray in acoustic mode field (local plateau)

- nonlinear ray - ion-acoustic interaction

⇒ analogous to nonlinear Landau damping

⋮

ongoing research topic.

→ Revisiting Langmuir Turbulence — $\left\{\begin{array}{l}\text{Collapse} \\ \text{Singularity} \\ \text{Formation} \\ \text{etc.}\end{array}\right.$

Recall:

① the 1,000,000 question of turbulence (N.-S.):

prove $\nabla V \to \infty$ as $r \to 0$, so ϵ finite as $r \to 0$

i.e. $\langle (\nabla V)^2 \rangle \sim \epsilon / r$

i.e. singularity formation lies at the heart of turbulence physics.

② Zakharov Equations — Langmuir Turbulence

$$i \frac{\partial \mathcal{E}}{\partial t} + \frac{3}{2} \omega_p \lambda_D^2 \nabla^2 \mathcal{E} = \frac{\omega_p}{2n_0} \delta n \, \mathcal{E}$$

Plasma wave envelope

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right) \delta n = \frac{1}{16\pi M_i} \nabla^2 |\mathcal{E}|^2$$

ion-acoustic wave

and in subsonic case → NLS

$$i \frac{\partial \mathcal{E}}{\partial t} + \frac{3}{2} \omega_p \lambda_D^2 \nabla^2 \mathcal{E} = -\frac{e^2}{4 m_e \omega_p T_e} |\mathcal{E}|^2 \mathcal{E}$$

$\left\{\begin{array}{l}\text{hereafter} \\ \text{the} \\ \text{focus....}\end{array}\right.$

③ linearized stability analysis $\Leftrightarrow$ cla! EM self-focusing

$\Rightarrow$ instability for $\underbrace{\frac{E_0}{nT} > (k\lambda_D)^2}_{\text{plasma carrier}}$ $\left\{ \begin{array}{l} \text{intensity} \\ \text{vs.} \\ \text{diffraction} \end{array} \right.$

④ in wave kinetics, interactions tend to remove population inversion $\Rightarrow$ form low-k condensate

Now nonlinear evolution $\Rightarrow$ is there a singularity ? (akin to focus)

– assume spherical symmetry, for simplicity $\quad \mu = m_e/m_i$

$$- \; r = \frac{3\lambda_D}{\sqrt{2\mu}}\rho, \quad t = \frac{3}{\mu}\frac{\tau}{\omega_p}, \quad \delta n = \frac{2}{3} n_0 \mu V$$

$$E = 4(2\pi n_0 T \mu)^{1/2} \phi$$

$$\boxed{i\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial \rho}\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\rho^2\phi + |\phi|^2\phi = 0} \qquad (*)$$

Now, (A) has two integrals of motion:

1) $I_1 = \int_0^\infty \rho^2 |\phi|^2 d\rho$ → plasmon #

2) $I_2 = \int_0^\infty \left\{ \left| \frac{\partial}{\partial\rho}(\rho\phi) \right|^2 + 2|\phi|^2 - \frac{1}{2}\rho^3|\phi|^4 \right\} d\rho$ → plasmon energy / Hamiltonian

if define variance of plasmon distribution:

$$A = \int d^3x \, (\Delta \underline{x})^2 |E|^2$$

$$= \int_0^\infty d\rho \, \rho^4 |\phi|^2 \geq 0$$

then straightforward algebra ⇒

$$\frac{d^2A}{dt^2} = 6I_2 - 2\int_0^\infty \left|\frac{\partial}{\partial\rho}(\rho\phi)\right|^2 d\rho - 4\int_0^\infty \rho^3|\phi|^4 d\rho < 6I_2$$

so $A < 3I_2 t^2 + C_1 t + C_2$

∴ $I_2 < 0$ ⇒ → $A \geq 0$ for only finite times

→ singularity (i.e. solution breaks down) at finite time.

$\Rightarrow$ Collapse$_2$, in finite time $\Leftrightarrow$ "finite time singularity"

$\therefore$ "caviton" formed $\Rightarrow$ can access any dissipation

Now:

- $I_2 < 0 \Rightarrow \dfrac{\mathcal{E}_0}{nT} > k^2\lambda_D^2$

- NLS has self similar solution

$$\phi = \exp\left\{ i\lambda^2 \int \frac{dt}{f(t)^2} \right\} \frac{1}{\lambda f(t)} R\left(\rho / \lambda f(t) \right)$$

$$\rho / \lambda f(t) = \xi \,, \quad f(t) = \beta (t_0 - t)^{1/2}$$

singularity

- collapse $\Rightarrow$ another self-similar route to dissipation

contrast: K41 cascade.